FUNDAMENTALS OF STRUCTURAL DYNAMICS

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- Topics :

- Revision of single degree-of freedom vibration theory
  - Response to sinusoidal excitation
  - Response to impulse loading
  - Response spectrum
  - Multi-degree of freedom structures

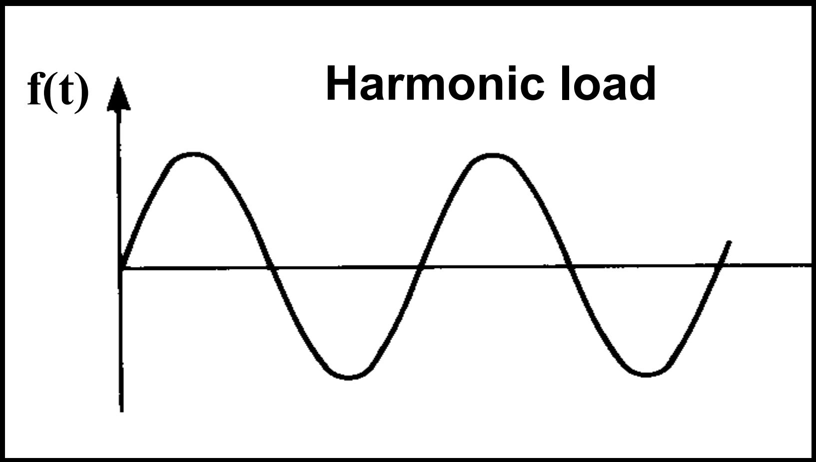
References :

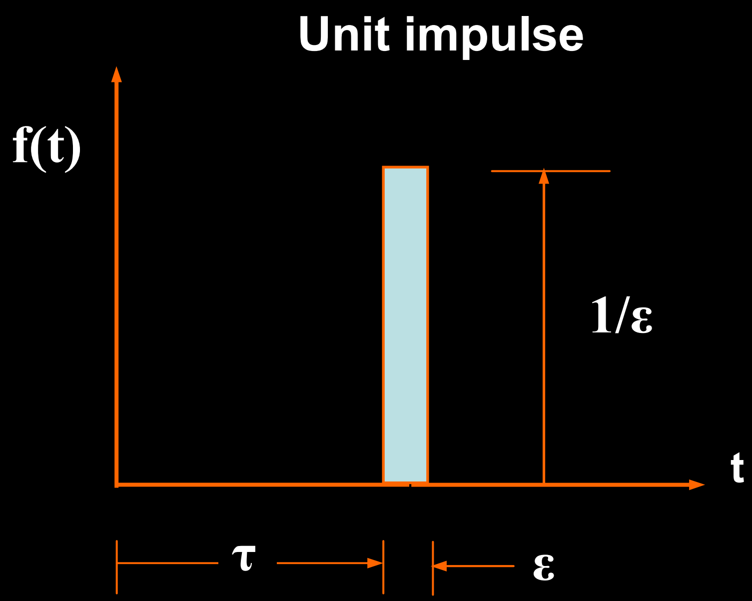
R.W. Clough and J. Penzien ‘Dynamics of Structures’ 1975

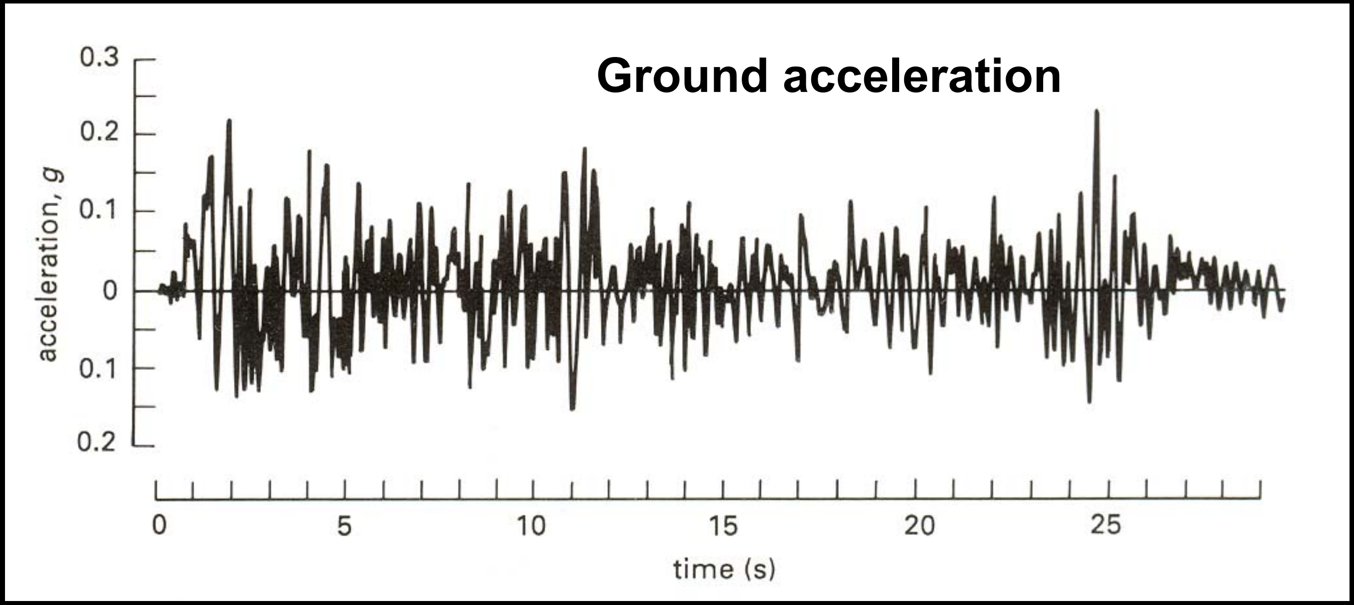
A.K.. Chopra ‘Dynamics of Structures: Theory and Applications to Earthquake Engineering’ 20011

G.D. Manolis, Analysis for Dynamic Loading, Chapter 2 in Dynamic Loading and Design of Structures, Edited by A.J. Kappos, Spon Press, London, pp. 31-65, 2001.

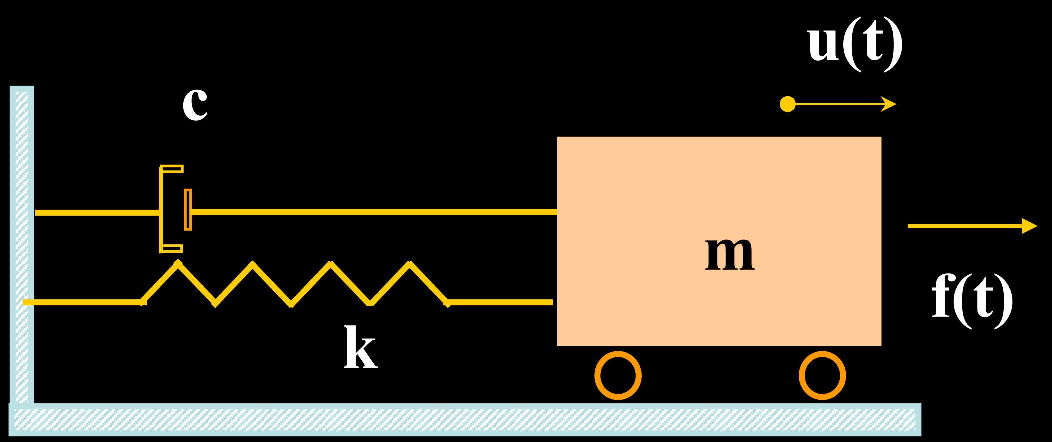
Why dynamic analysis? → Loads change with time







Single degree of freedom (sdof) system



mass-spring-damper system

Mass m (kgr, tn), spring parameter k (kN/m), viscous damper parameter c (kN*sec/m), displacement u(t) (m), excitation f(t) (kN).

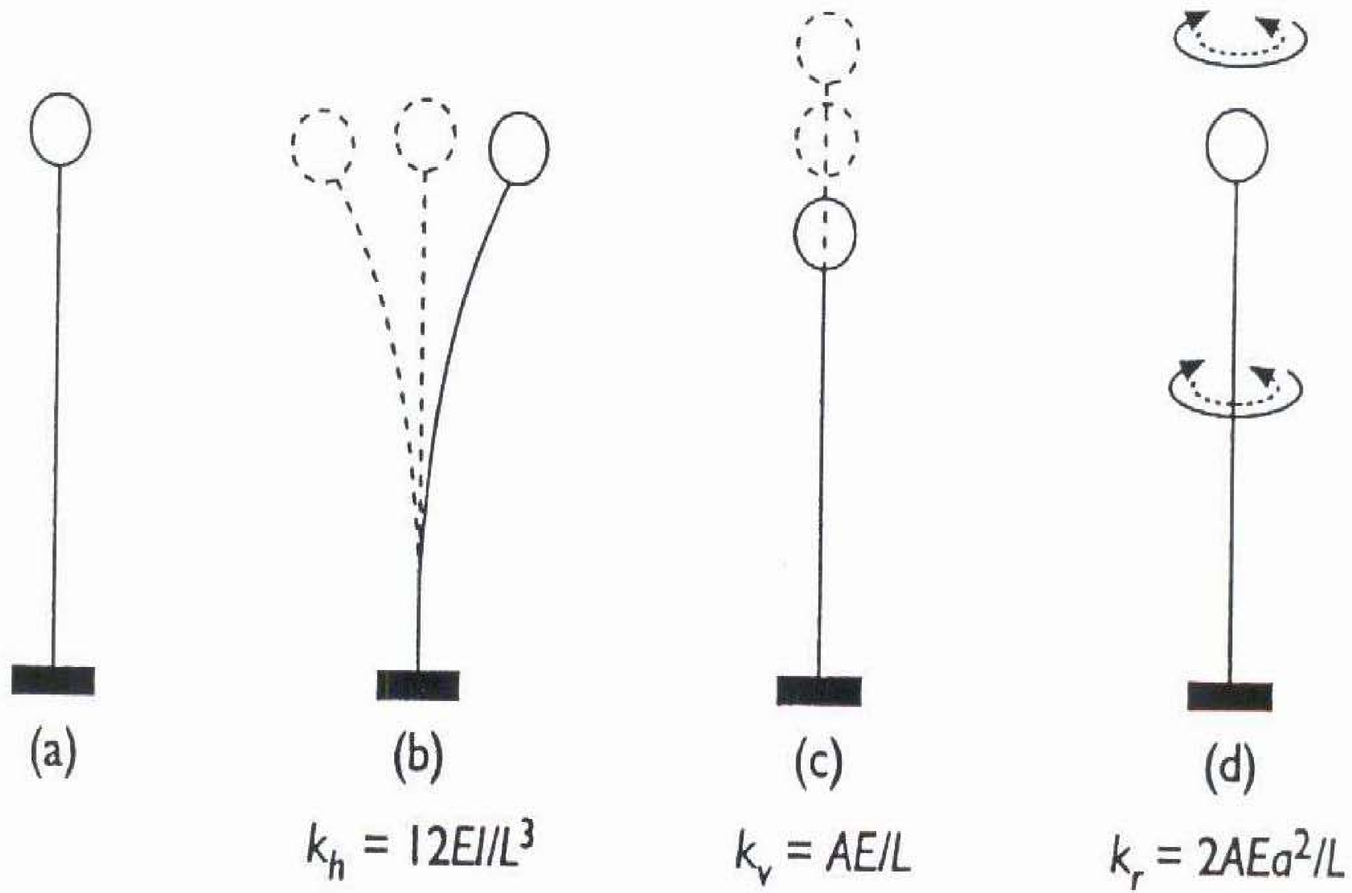


Figure 2.1 (a) SDOF modelling of a single story frame for (b) horizontal, (c) vertical and (d) rotational oscillations.

Definitions of restoring force parameter k

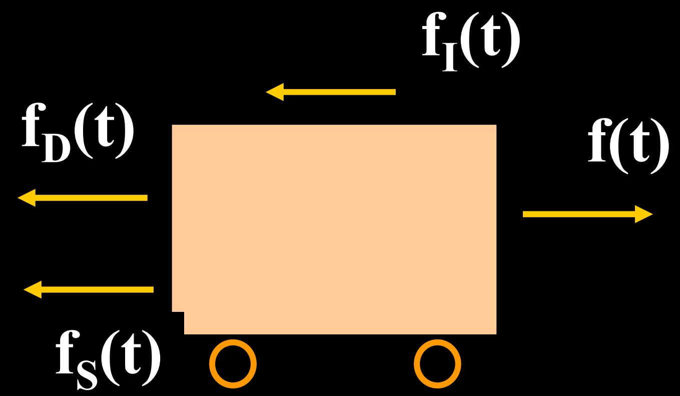
Dynamic equilibrium – D'Alembert's principle

$f(t) = f_I(t) + f_D(t) + f_S(t)$

Inertia force $f_I(t)$,

Damping force $f_D(t)$

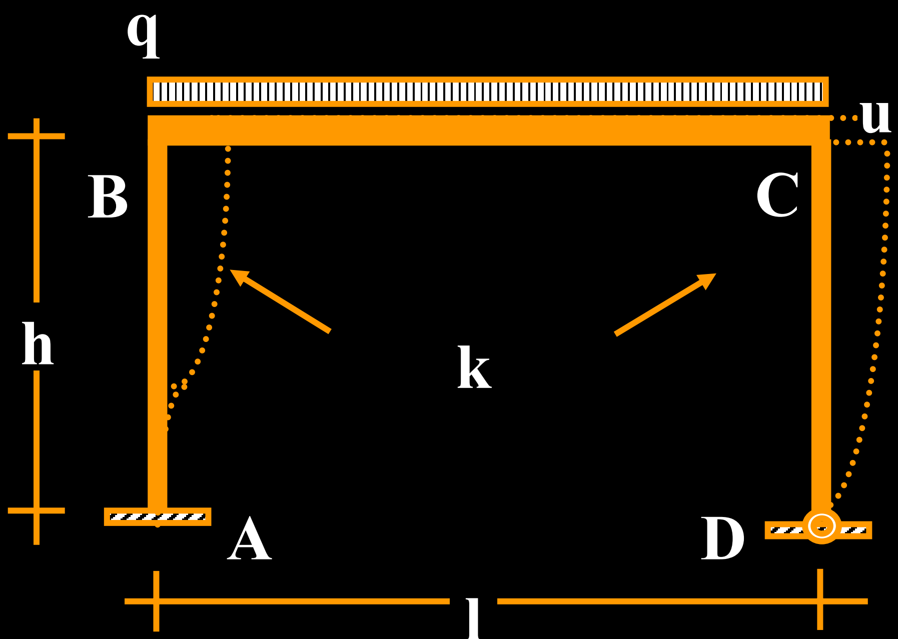
Restoring (elastic) force $f_S(t)$



Setting response parameters as: displacement u(t) (in m), velocity u'(t) (in m/s) and acceleration u''(t) (in m/s^2), then:

$f_I(t) = m\,u''(t)$, $f_D(t) = c\,u'(t)$, $f_S(t) = k\,u(t)$.

Shear plane frame - dynamic parameters



Rigid beam, mass less columns. Total weight (mass) accumulated in the middle of the beam.

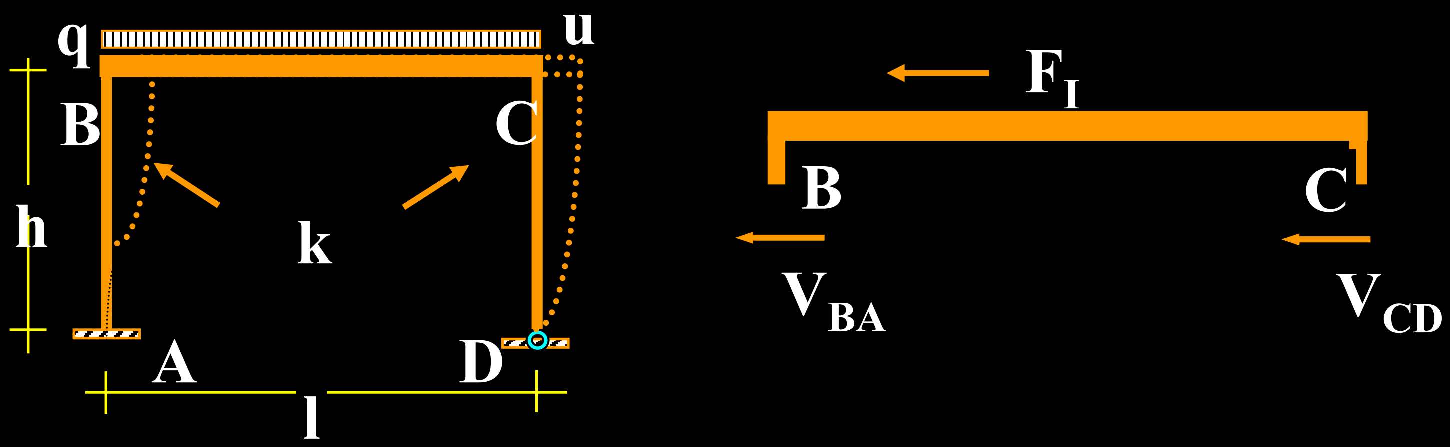
AB – Fixed end

CD – Hinged end

$$m = w/g = (ql)/g$$

$$k = f_{st}(u{=}1) = V_{BA} + V_{\Gamma\Delta} = 12EI/h^3 + 3EI/h^3 = 15EI/h^3$$

Free vibration with no damping



No external force f(t). Oscillations due to initial conditions at t = 0. Initial displacement u_0 or/and initial velocity u'_0

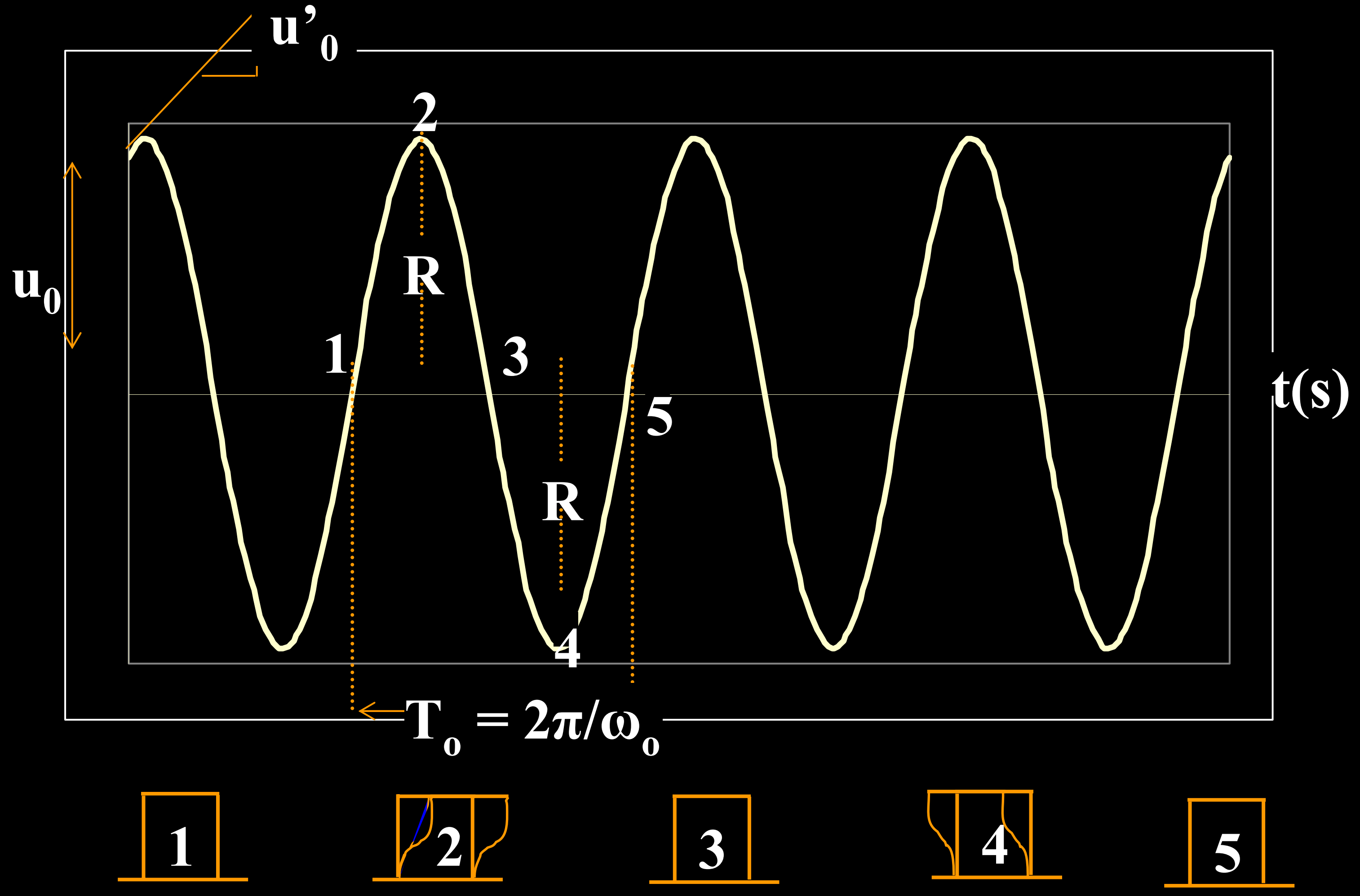
$$m\,u''(t) + k\,u(t) = 0$$

$$u(t) = R_1 \sin \omega t + R_2 \cos \omega t = R \sin(\omega t+\theta)$$

where $R^2 = R_1^2 + R_2^2$ and $\tan \theta = R_2/R_1$

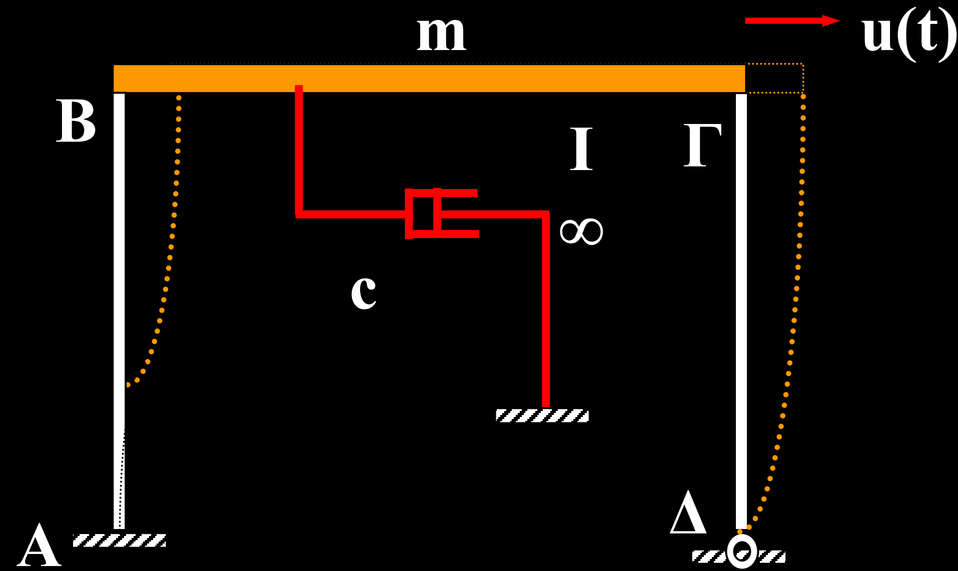
Natural frequency $\omega = [k/m]^{1/2}$ (rad/s),

Nat. period $T = 2\pi/\omega$ (sec)



Unrealistic – no decay

Free vibration with damping



Equation of motion → Homogeneous 2nd order-ODE:

$$m\,u''(t) + c\,u'(t) + k\,u(t) = 0$$

Characteristic equation $$(mr^2 + cr + k) = 0$$

and roots: $$r_{1,2} = \pm\sqrt{\frac{c^2}{(2m)^2} - \frac{k}{m}}$$

$$\frac{c^2}{(2m)^2} - \frac{k}{m} \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

< 0 ↓ **oscillation**

$[c/2m]^2 - k/m = 0$ ➔

$c_{cr} = 2\sqrt{k * m} = 2m\omega_0$

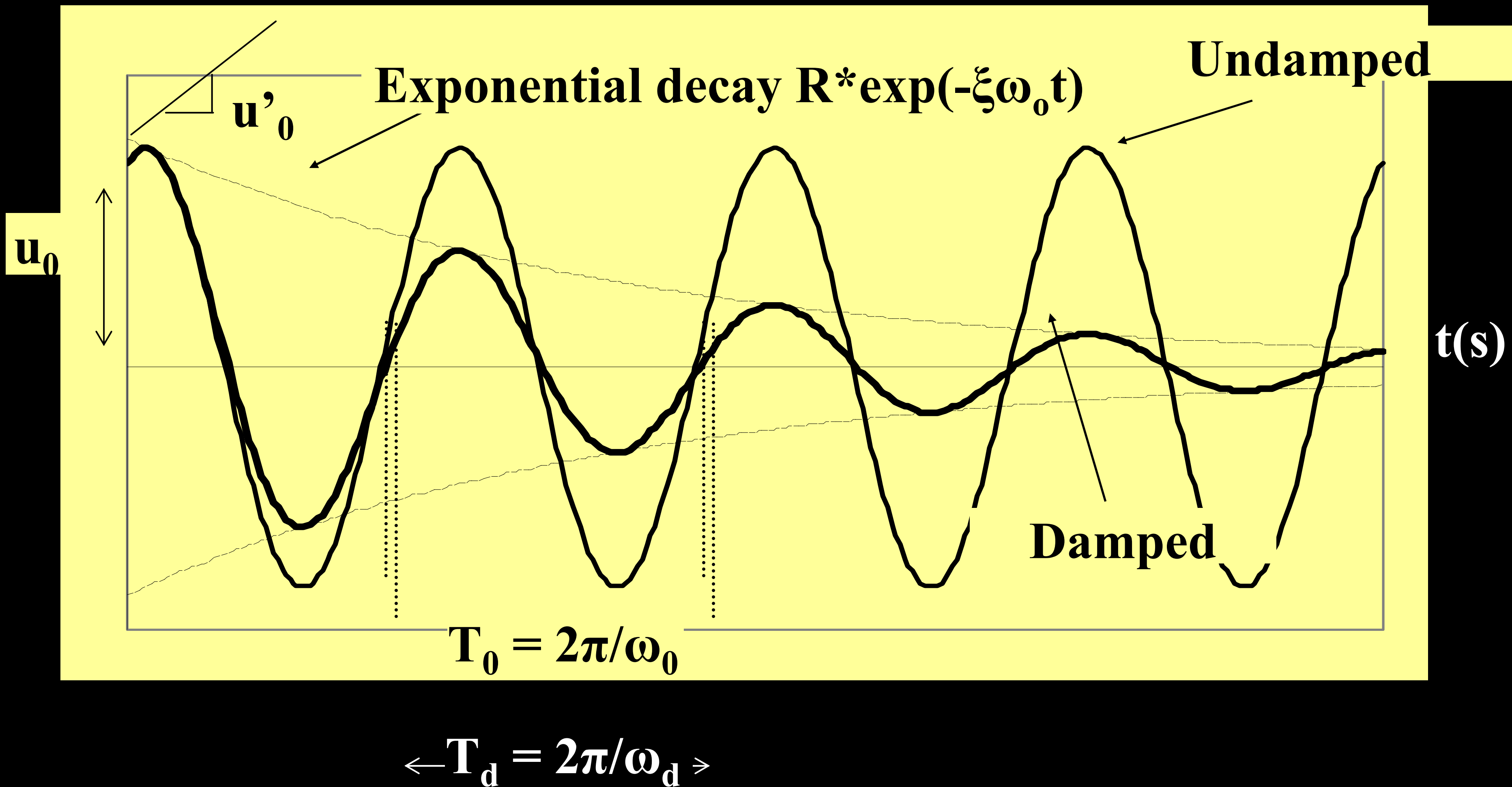
c_{cr} = critical damping

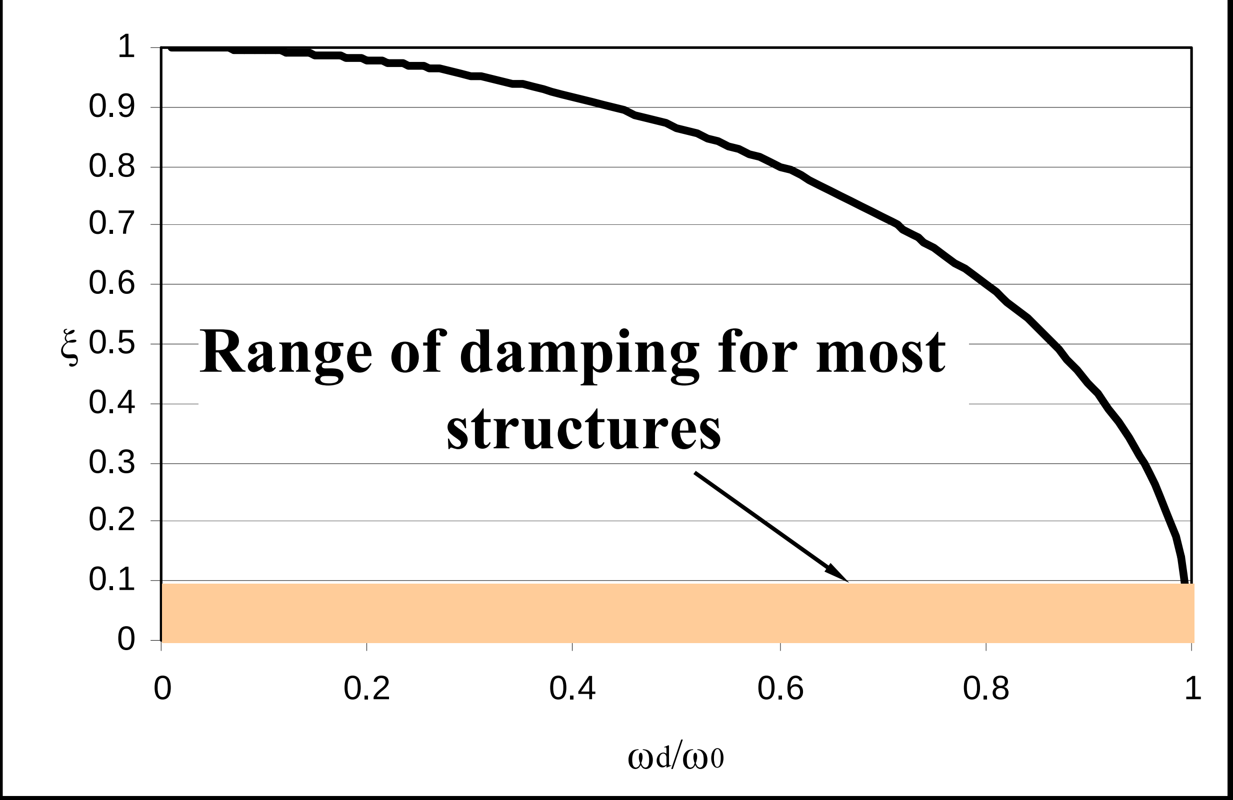
Critical damping ratio $\xi = \dfrac{c}{c_{cr}} = \dfrac{c}{2m\omega_0}$

For $\xi < 1.0$**, and setting** $\omega_d = \omega_0\sqrt{1 - \xi^2}$

$$u(t) = e^{-\xi\omega_0 t}(R_1 \sin \omega_d t + R_2 \cos \omega_d t) = R\, e^{-\xi\omega_0 t} \sin(\omega_d t + \theta)$$

$$R_1 = \frac{\dot{u}_0 + u_0\xi\omega_0}{\omega_d}, \quad R_2 = u_0, \quad R = \sqrt{R_1^2 + R_2^2}, \quad \tan\theta = \frac{R_2}{R_1}$$


Exponential decay R*exp(-ξω_ot)
Undamped
u'_0
u_0
t(s)
Damped
T_0 = 2π/ω_0
T_d = 2π/ω_d



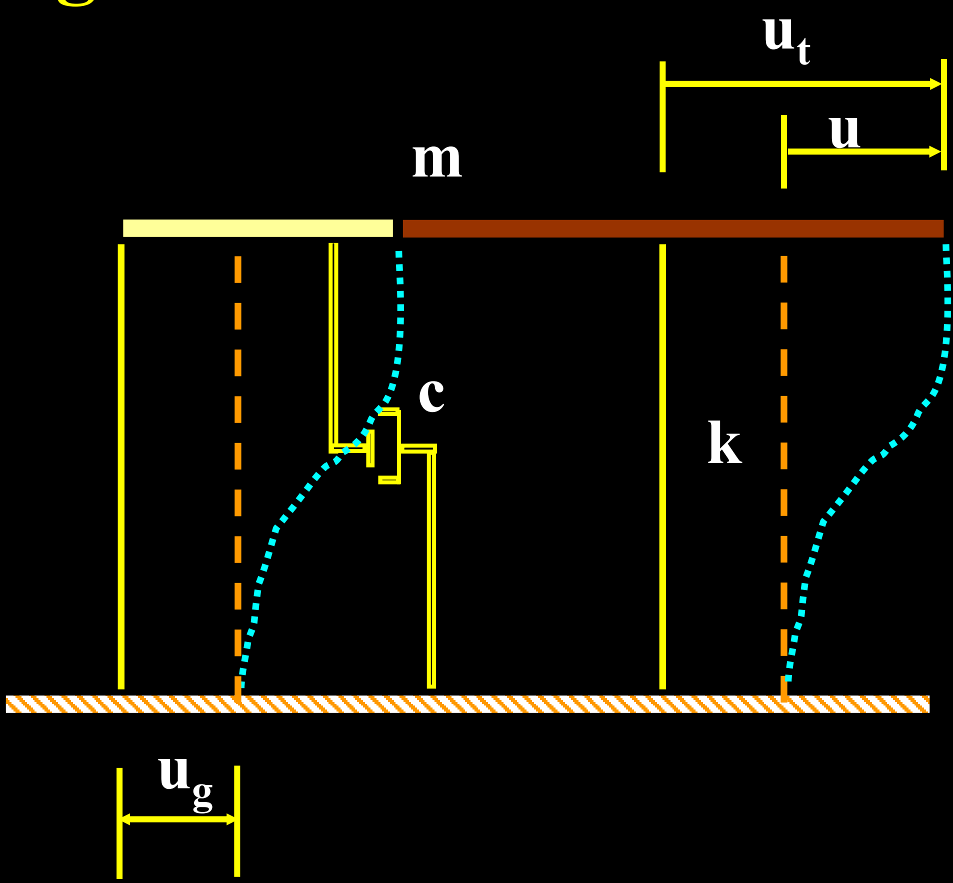
Logarithmic decrement $\delta = 2\pi\xi$, relates the magnitude of successive peaks

$$\ln(R_j/R_{j+n}) = n\,\frac{2\pi\xi}{\sqrt{1-\xi^2}} \approx n*2\pi\xi = n\delta$$

Oscillation due to ground motion

Total displacement (u_t), ground displacement (u_g), relative displacement (u).

$$u_t(t) = u_g(t) + u(t)$$



Dynamic equilibrium:

$$f_I + f_D + f_S = 0$$

$f_I = m\, u_t''(t)$ $\quad$ $f_D = c\, u'(t)$ $\quad$ $f_S = k\, u(t)$

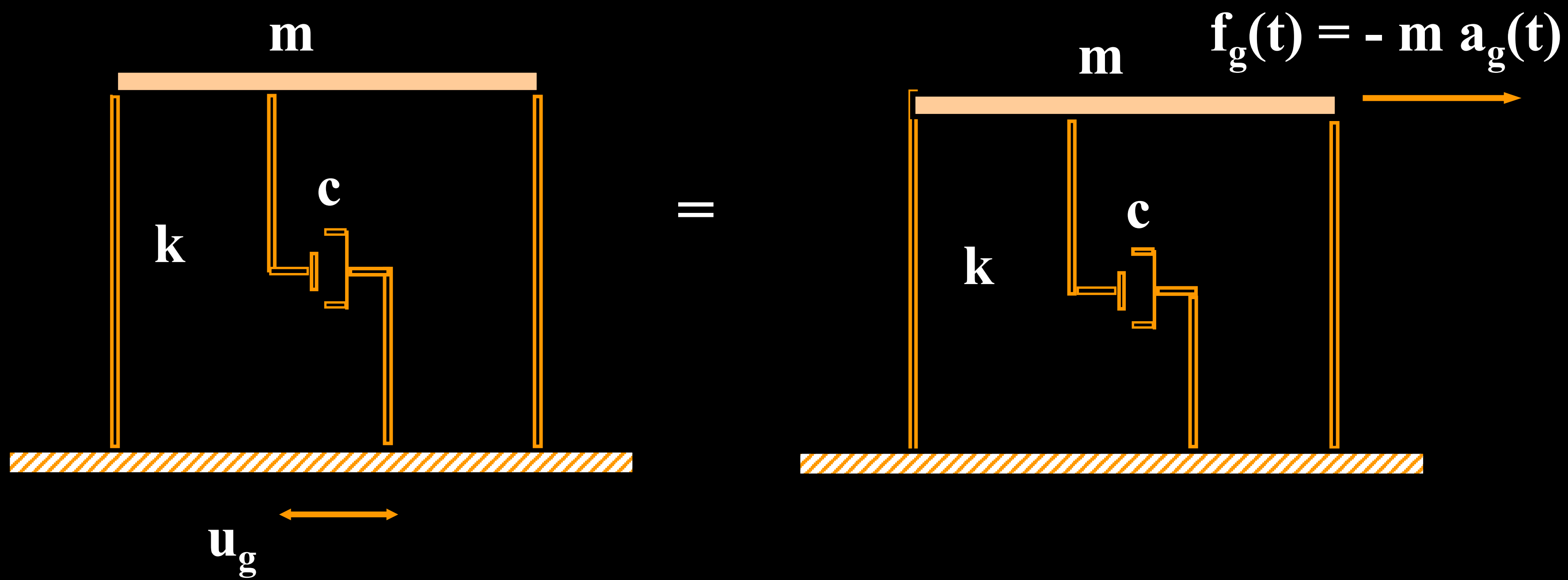
Equation of motion:

$$m\, u_t''(t) + c\, u'(t) + k\, u(t) = 0$$

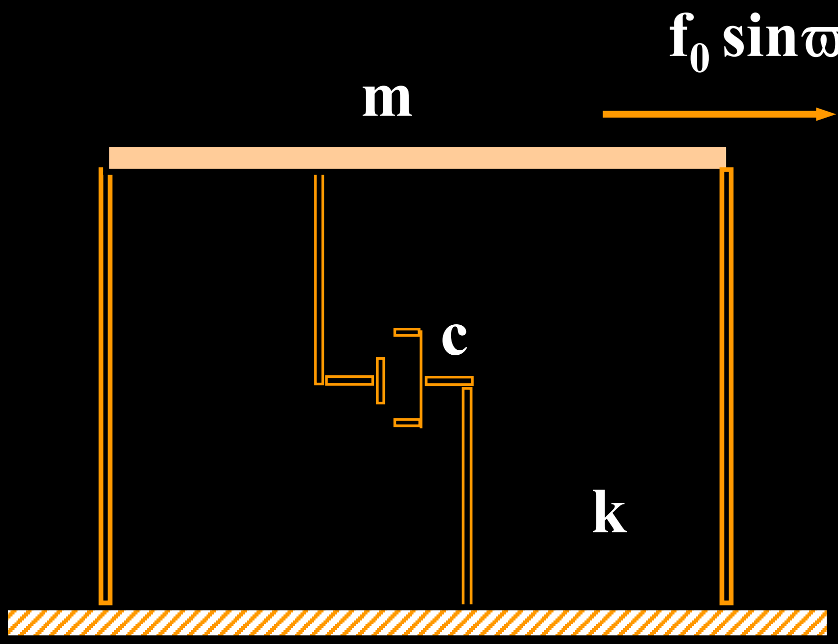
Setting $u_t''(t) = a_g(t) + u''(t)$, where $a_g(t)$ = ground acceleration, the equation of motion becomes:

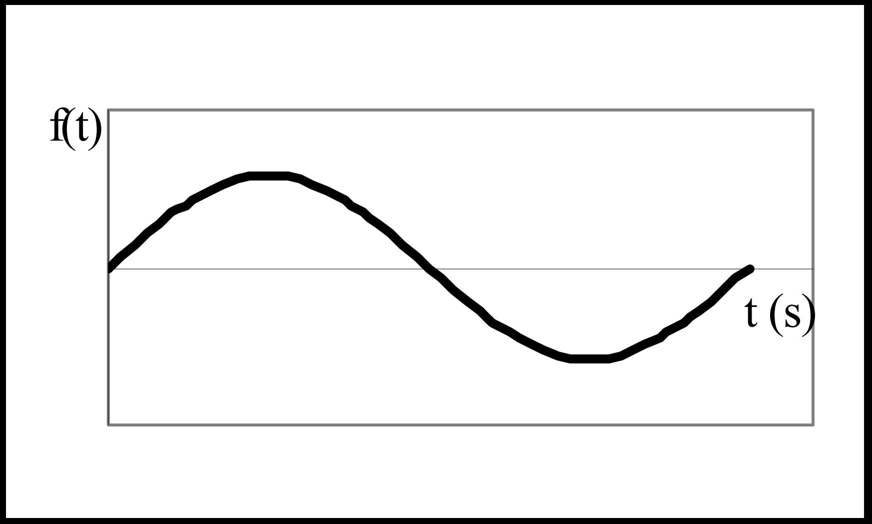
$$m\,u''(t) + c\,u'(t) + k\,u(t) = -\,m\,a_g(t) = f_g(t)$$

The above is the equation of motion of a fixed-base frame under an external dynamic force $f_g(t)$.



Harmonic excitation





Force with amplitude f_0 and excitation frequency $\overline{\omega}$

Equation of motion → Non-homogeneous 2nd order-ODE:

$$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) = f_0 \sin \overline{\omega}\, t.$$

Two part solution → $u(t) = u_c(t) + u_p(t)$

Complementary component (transient)

$$u_c(t) = e^{-\xi\omega_0 t}\,(C_1 \sin \omega_d t + C_2 \cos \omega_d t)$$

Particular component (steady-state)

$$u_p(t) = \frac{f_0}{k}\,\frac{1}{\sqrt{(1-\beta^2)^2 + (2*\beta\xi)^2}} * \sin(\overline{\omega}\,t-\theta) = \rho \sin(\overline{\omega}\,t-\theta)$$

where $\beta = \dfrac{\overline{\omega}}{\omega_0}$ = frequency ratio

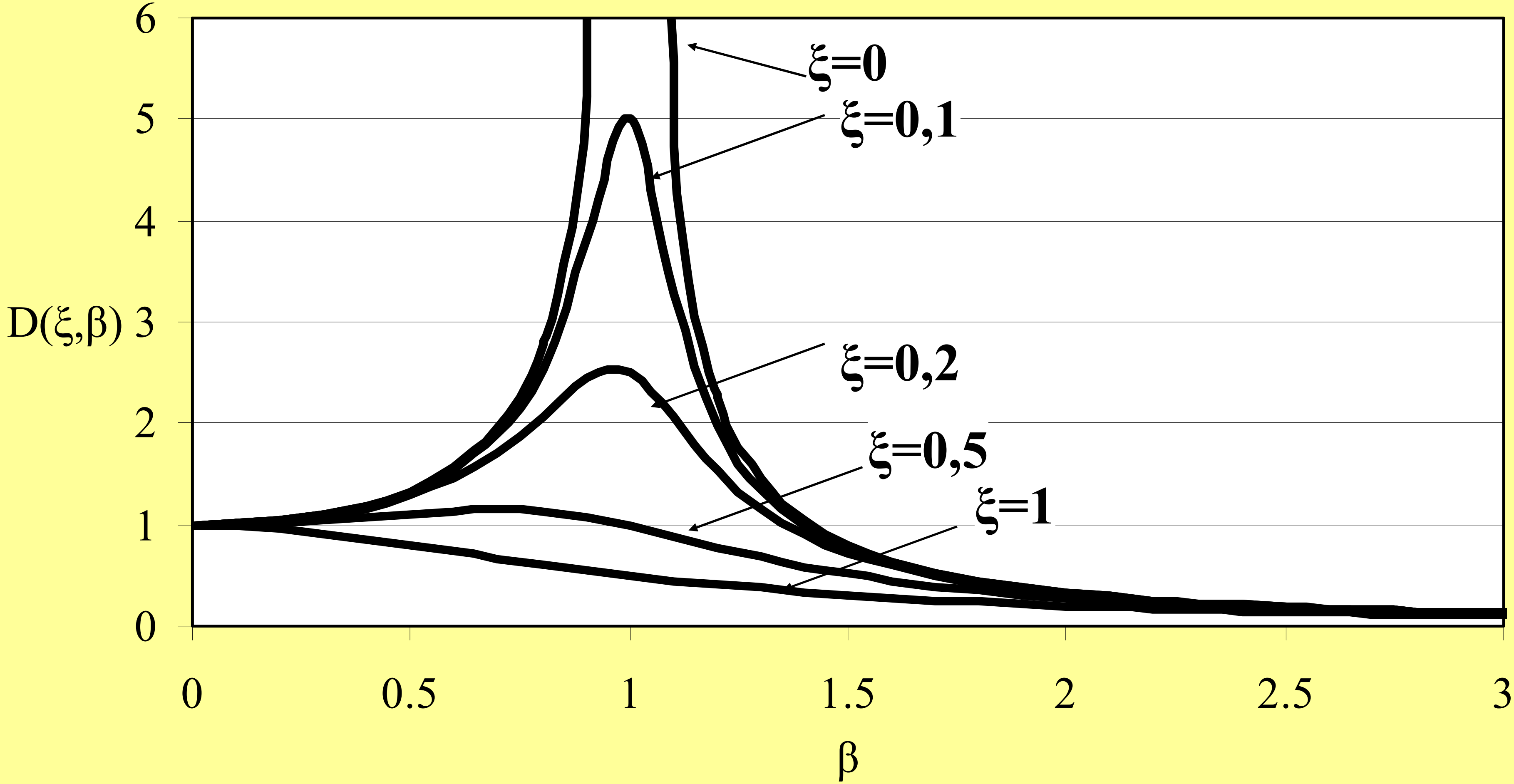
Phase θ is determined via the relation: $\tan\theta = \dfrac{2\xi\beta}{1-\beta^2}$

The steady-state peak ρ is related to the peak of the static response u_{st} (corresponding to static force $f_{st} = f_0$).

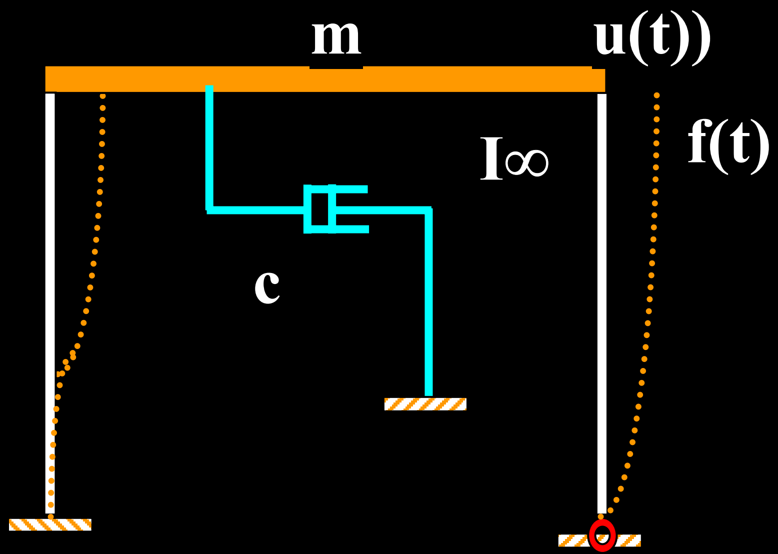
$$\rho = \frac{f_0}{k} D(\beta,\xi) = u_{st} D(\beta,\xi)$$

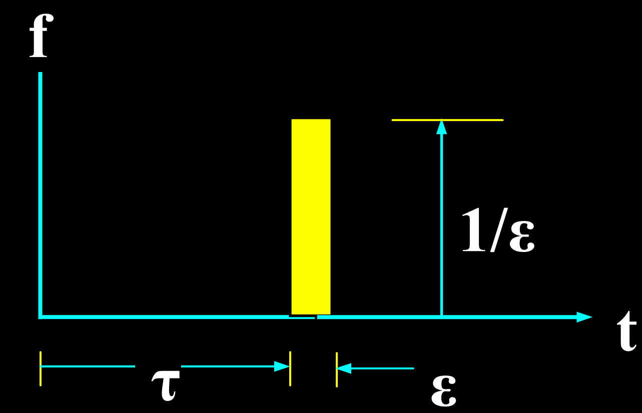
Dynamic amplification factor D(β,ξ), expresses the degree of error, if an 'equivalent' static (instead of fully dynamic) analysis is performed

$$\mathbf{D(\beta,\xi)} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2*\beta\xi)^2}}$$



Unit impulse excitation

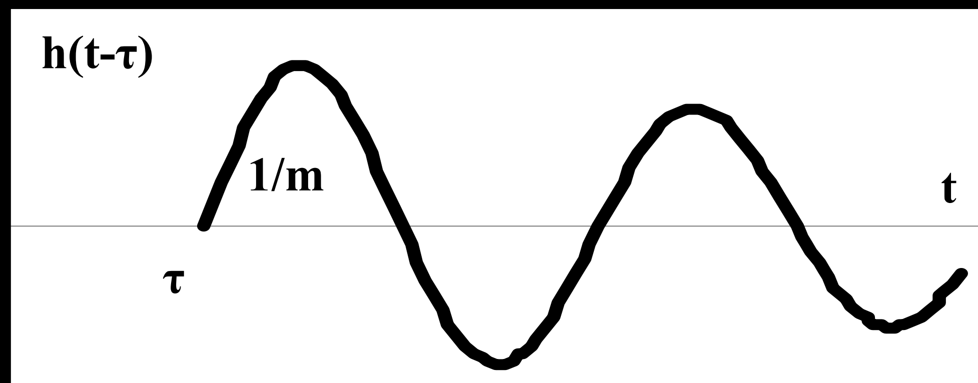




Due to infinitesimal duration ε, during impulse damping and restoring forces are not activated. After impulse, the system performs a damped free vibration with initial conditions u(τ) = 0, u'(τ) = 1/m, (change of momentum equal to applied force).

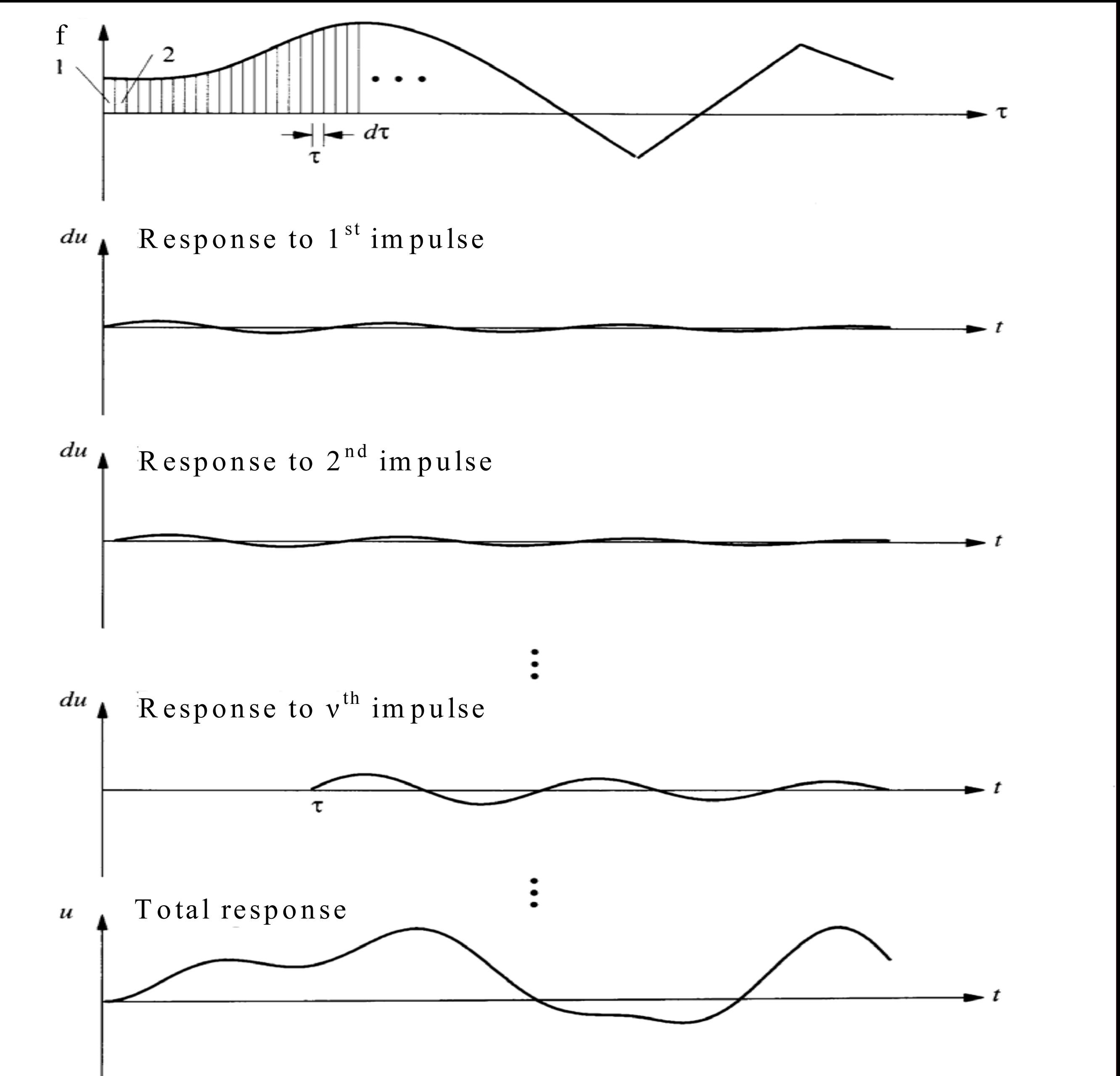
Unit impulse response function h(t-τ):

$$u(t) = h(t-\tau) = \frac{1}{m\omega_d} e^{-\xi\omega(t-\tau)} \sin[\omega_d(t-\tau)]$$



An impulse occurring at time τ, determines the response at a later time (t ≥ τ). Due to damping, the influence of an impulse weakens as the time interval increases (memory of vibration).

Response to arbitrary excitation

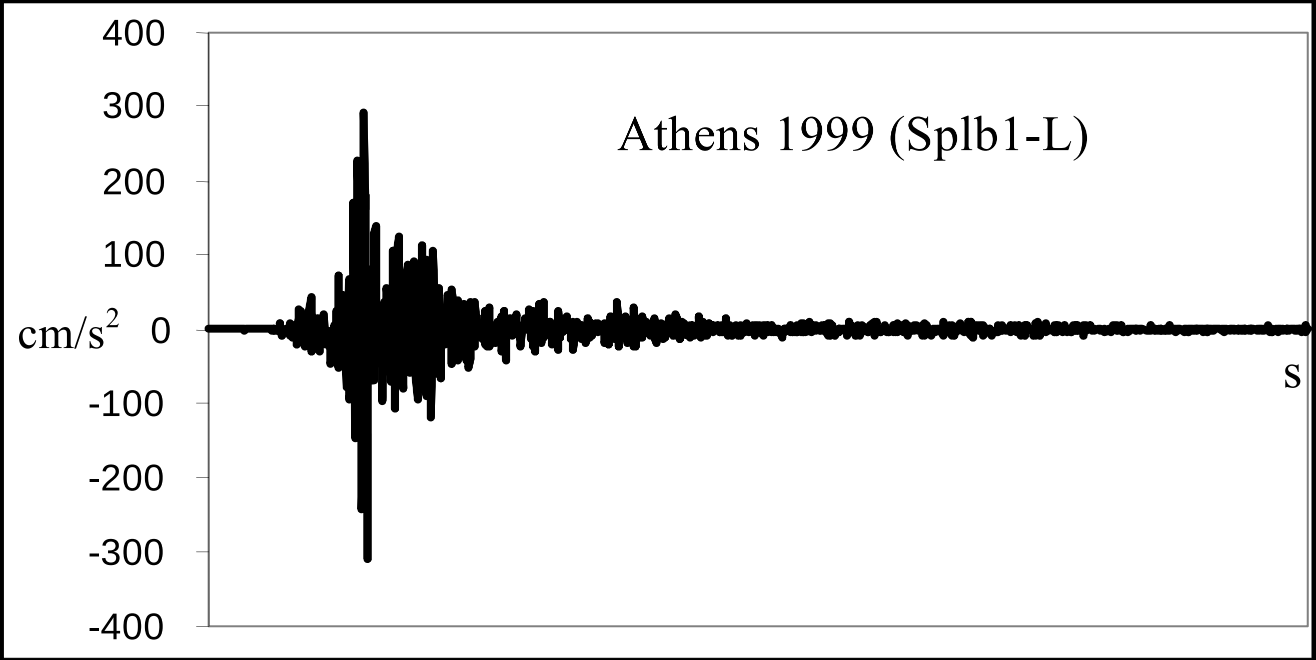


In the limit, for infinitesimal time steps, the summation of impulse responses becomes an integral - known as Duhamel's integral:

$$\mathbf{u(t)} = \int_0^t h(t-\tau)\, f(\tau)\, d\tau = \frac{1}{m\,\omega_d} \int_0^t \mathbf{f(\tau)\, e^{-\xi\omega o(t-\tau)} \sin[\omega_d(t-\tau)]d\tau}$$

The above relation provides a means for determination of the response of a single degree elastic system subjected to arbitrary excitation (in analytical or digital form).

Earthquake response spectra



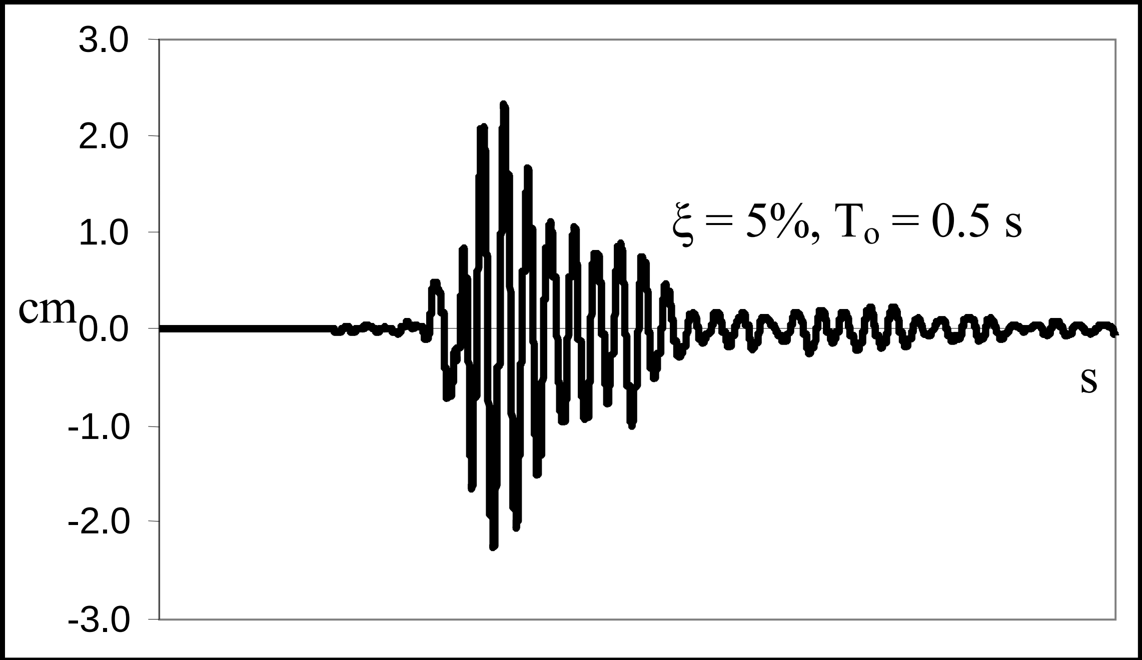
Equation of motion

$$m\,u''(t) + c\,u'(t) + k\,u(t) = -\,m\,a_g(t) = f_g(t)$$

Duhamel

$$y(t) = \int_0^t h(t-\tau)\, f_g(\tau)\, d\tau = \frac{1}{\omega_d} \int_0^t a_g(\tau)\, e^{-\xi \omega o(t-\tau)} \sin[\omega_d(t-\tau)]\, d\tau$$

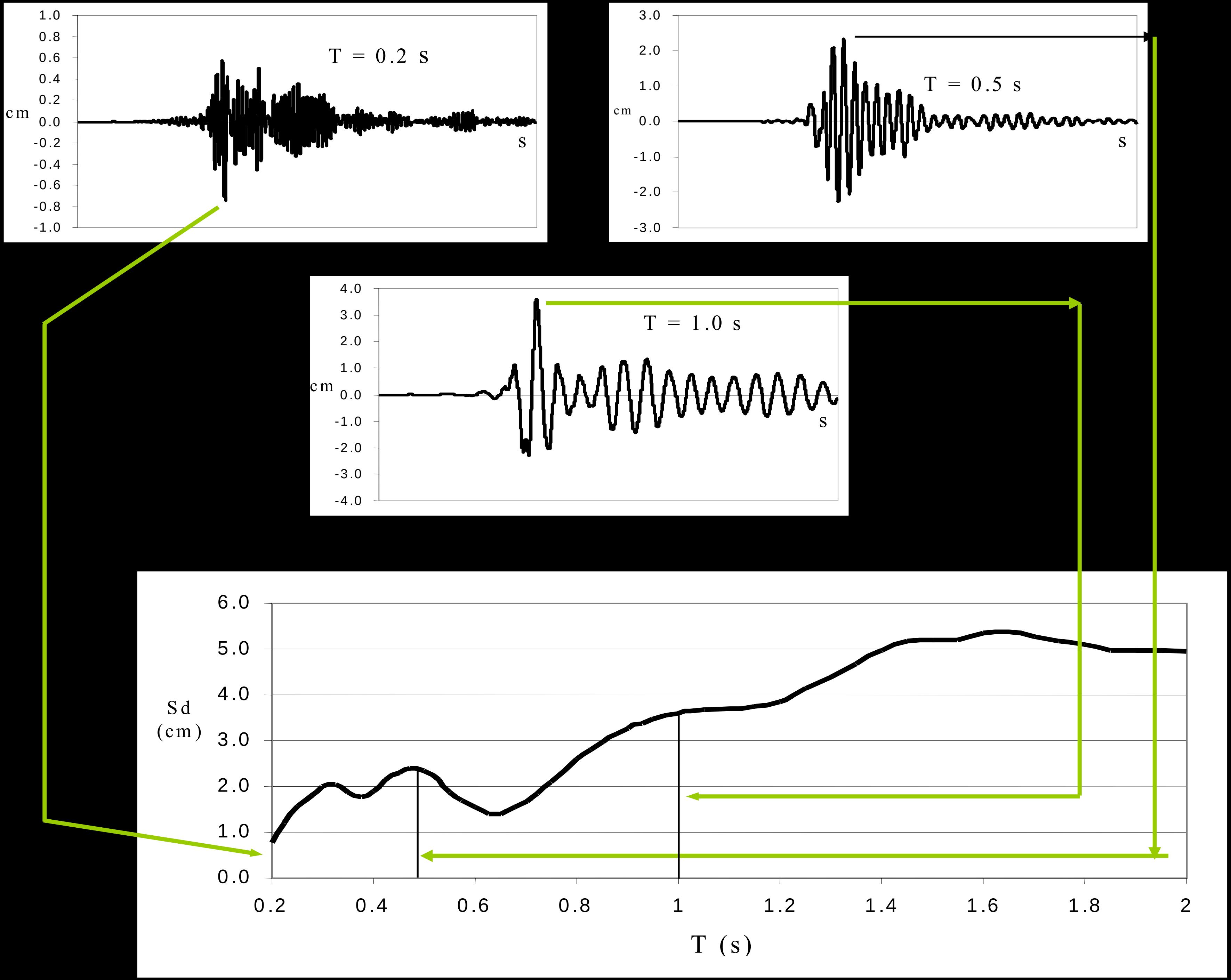
For a system with $\xi = 5\%$ και $T_o = 0.5$ s ($\omega_o = 12.57$ rad/s) the response was computed as →

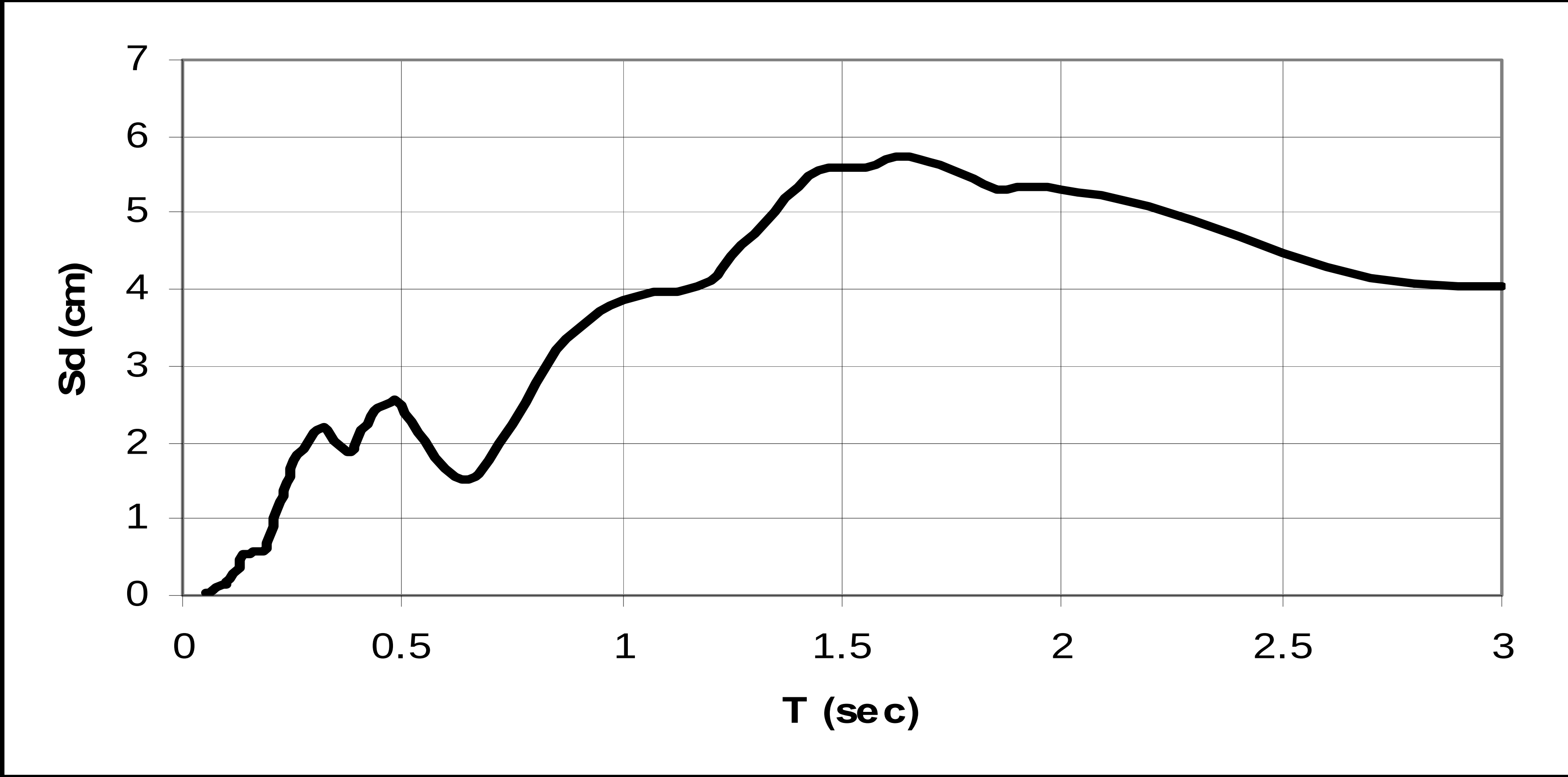


Quasi-harmonic response

For design purposes, only peak response parameters (displacement, velocity, acceleration, moments, shear forces) are of interest. These peak values, express the seismic **demand**.

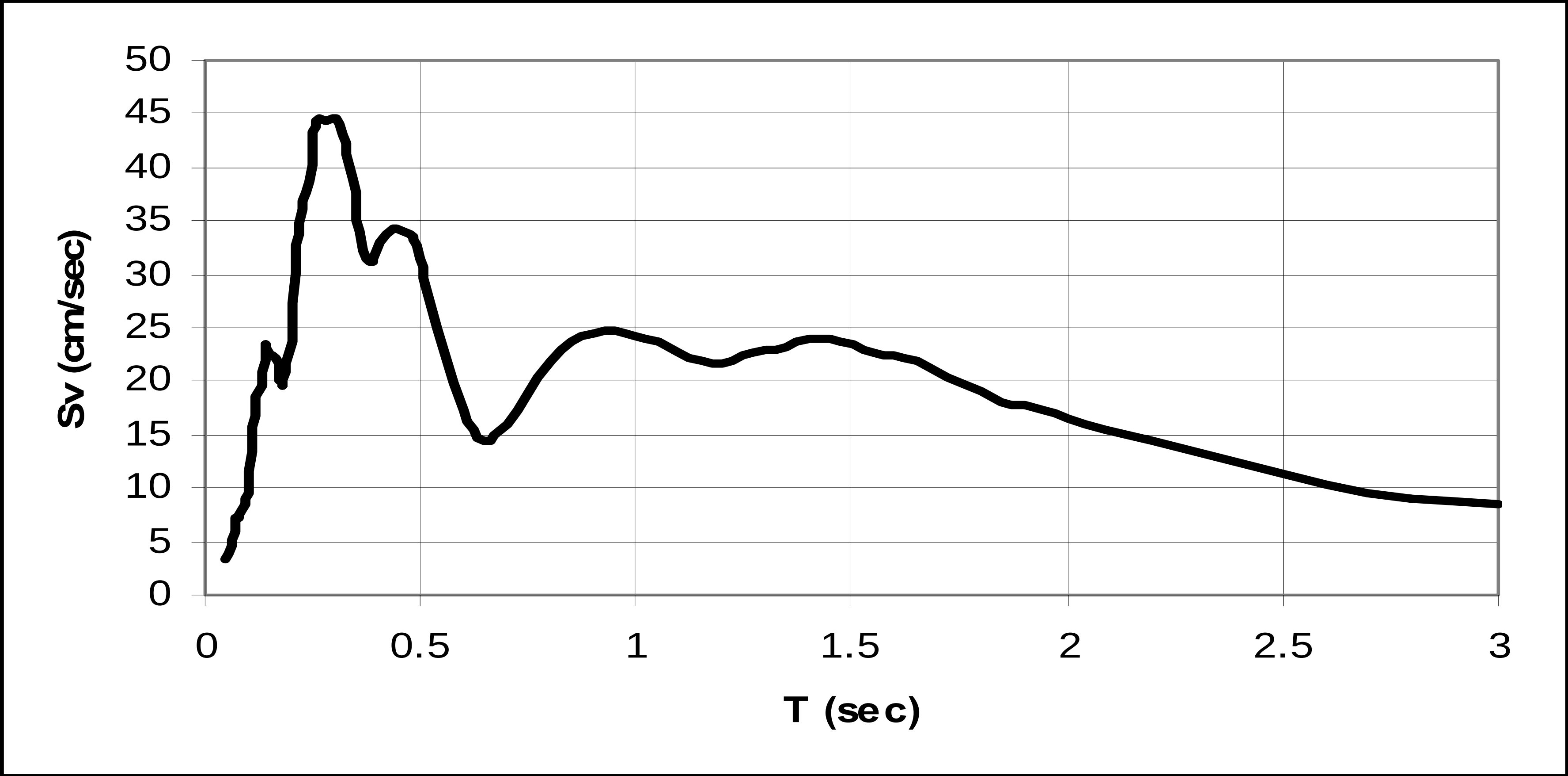
The seismic **demand** for systems with different periods is expressed via the **response spectra**.


T = 0.2 s
T = 0.5 s
T = 1.0 s
cm
s
Sd (cm)
T (s)



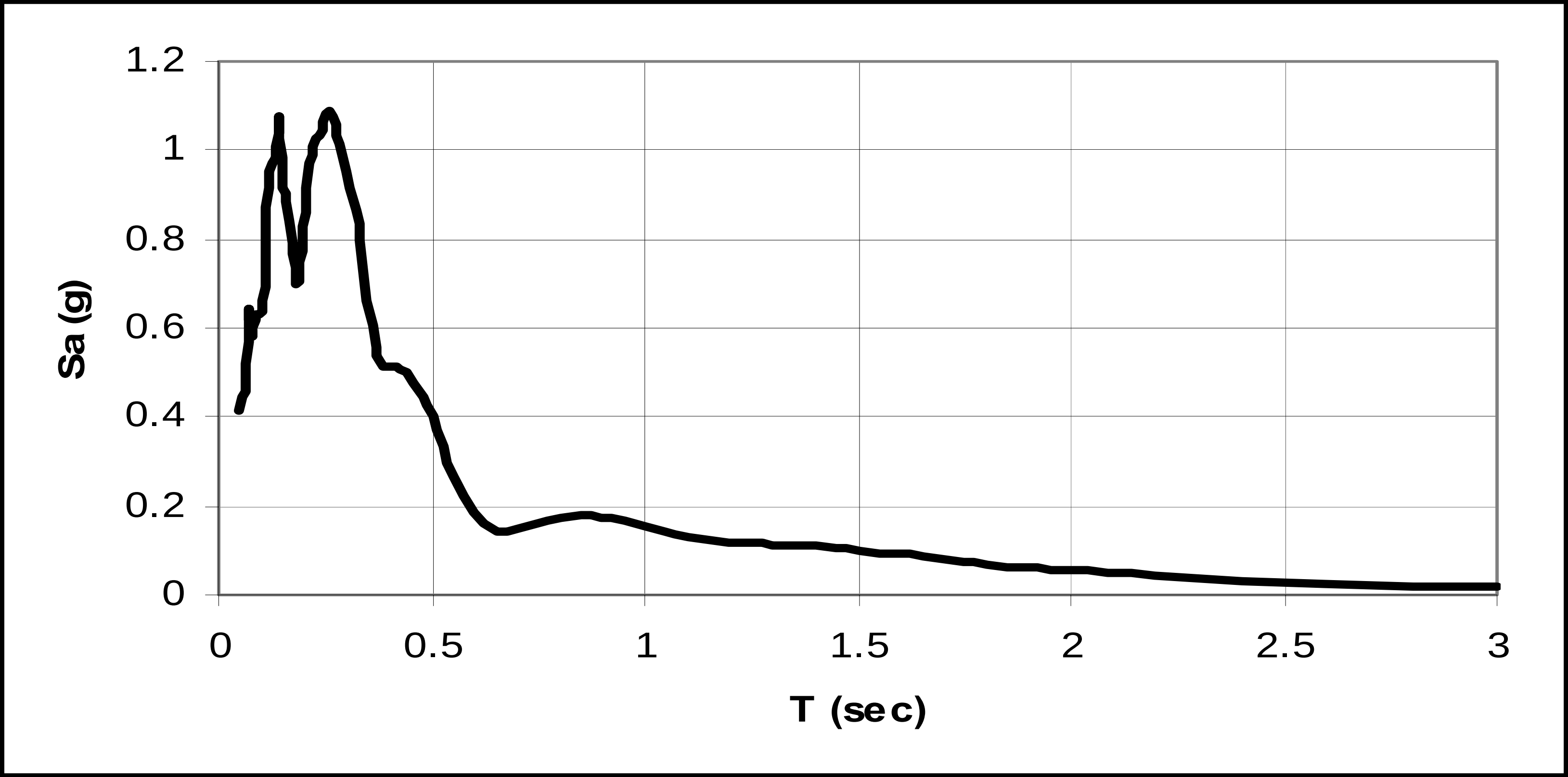
Displacement response spectrum S_d (Athens 99, component SPLB1-L).

The peak displacement values tend to increase with period (more flexible or taller structures, exhibit larger deflections).



Velocity response spectrum S_v

The previously noticed trend is not observed in S_v. After an initial rise, follows a relatively constant value range and then a decrease for large periods.



Acceleration response spectrum S_a

Here, an initial increase of S_a is followed by a rapid decrease for periods above 0.4 sec. (Flexible structures do not oscillate rapidly → small values of acceleration).

Actual shape depends on rapture characteristics and local soil conditions

If it is assumed that the response is quasi-harmonic with frequency equal to the natural frequency, then:

$$u(t) = u_{max}\sin\omega t,\ u'(t) = u_{max}\omega\cos\omega t,\ u''(t) = -\ u_{max}\omega^2\sin\omega t$$

Therefore, the following (approximate) relations between response spectra are often implemented:

$$S_v \approx \omega_0 * S_d = P\,S_v, \qquad S_a \approx \omega_0^2 * S_d = PS_a$$

where, $P\ S_v$ = pseudo-spectral velocity and PS_a = pseudo-spectral acceleration

These approximate relations enable us to present all 3 response spectra with one tri-partite logarithmic plot.

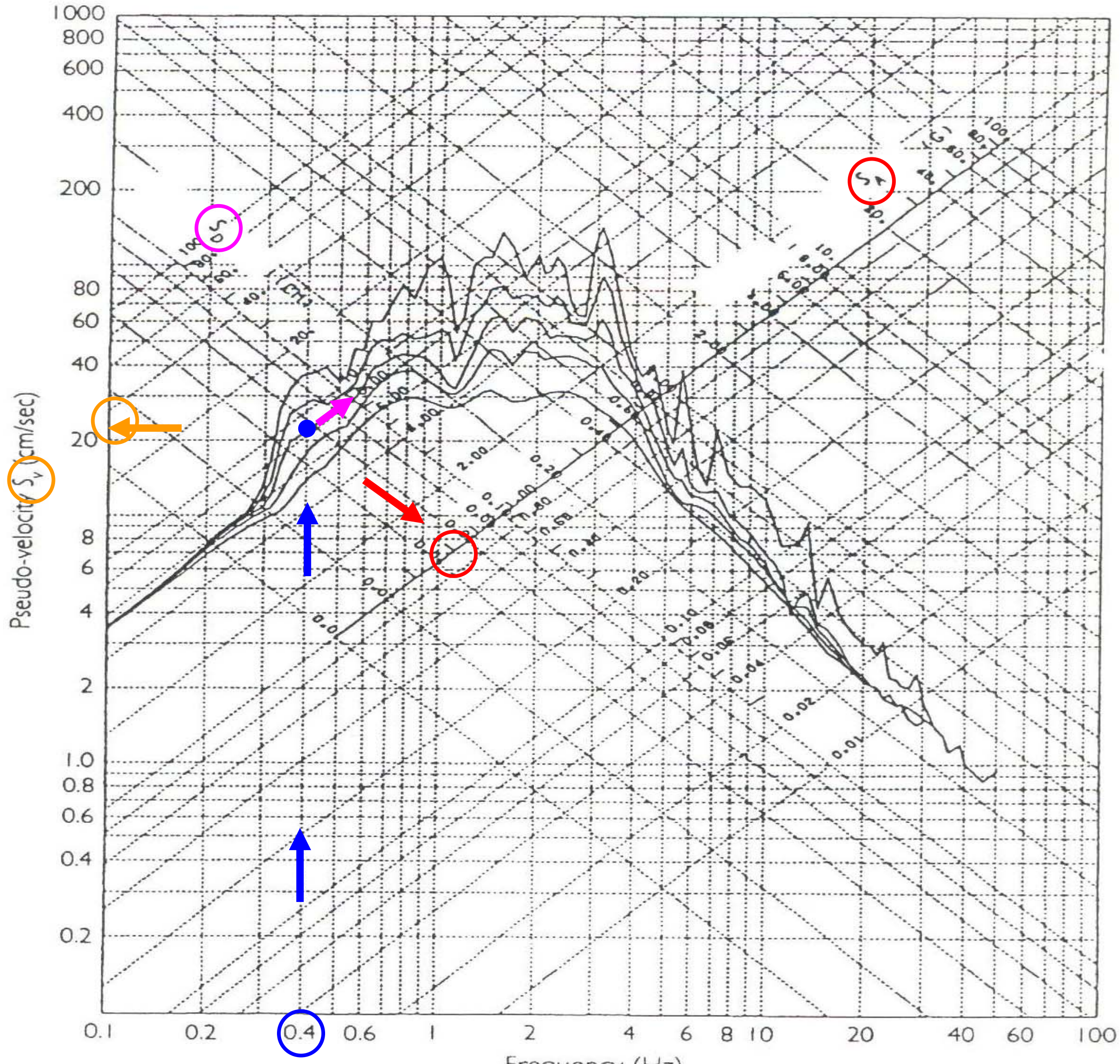
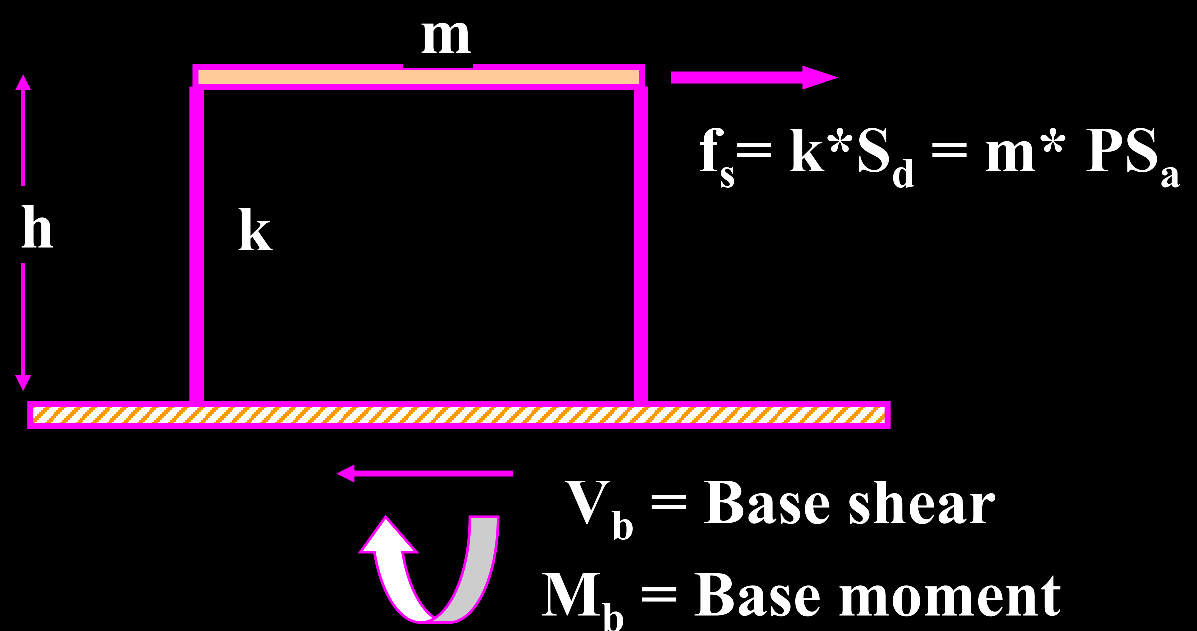


Figure 2.29 Triple spectrum for the Kalamata, Greece 1986 earthquake: velocity (cm/sec) along vertical axis; acceleration (g) along left to right axis; relative displacement (cm) along right to left axis; all versus frequency (Hz). Note: the five curves are for 0%, 2%, 5%, 10% and 20% damping.

Design parameters of response spectra

Static equivalence approach

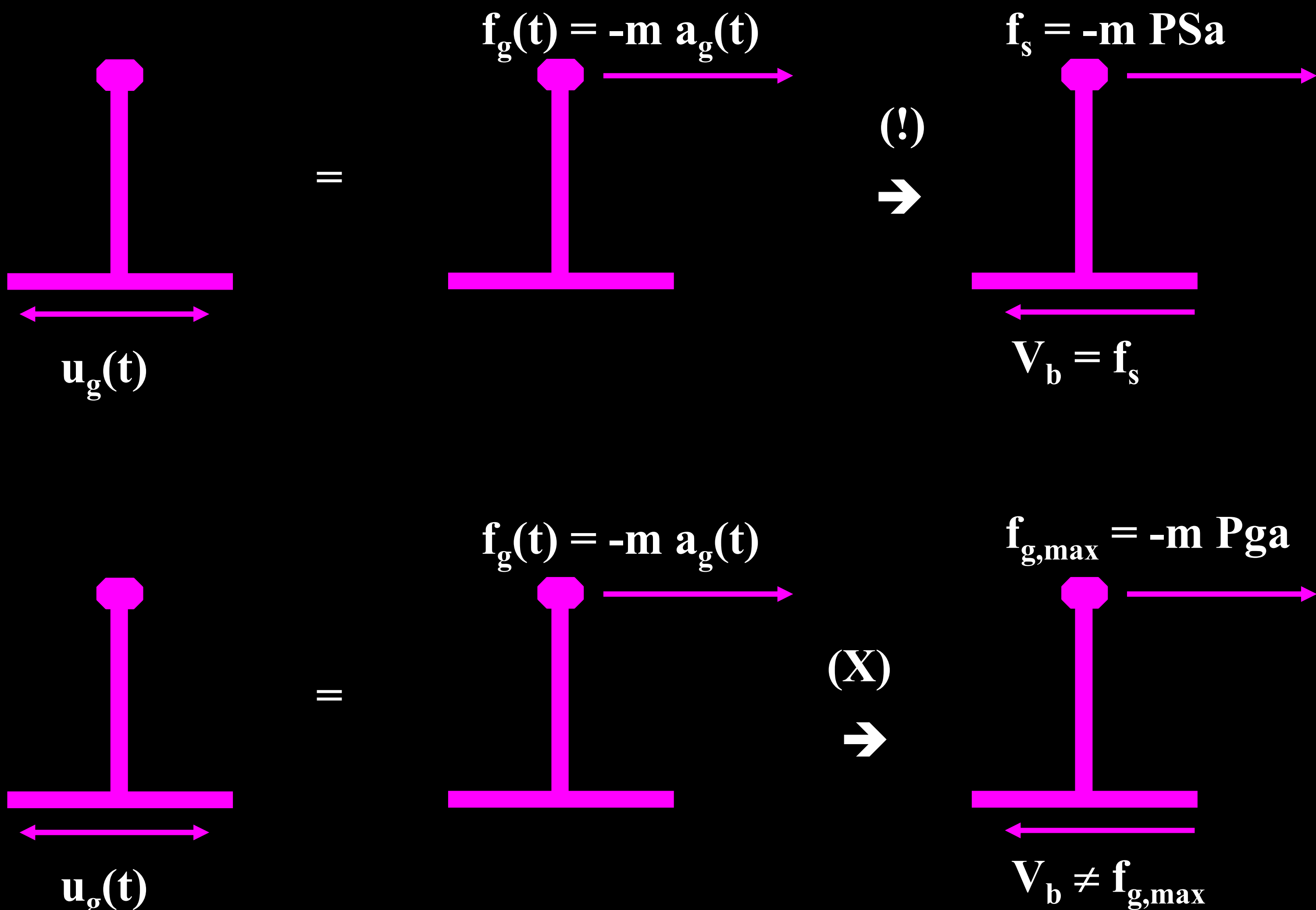


$$V_b = f_s = k*S_d$$
$$M_b = h*V_b$$

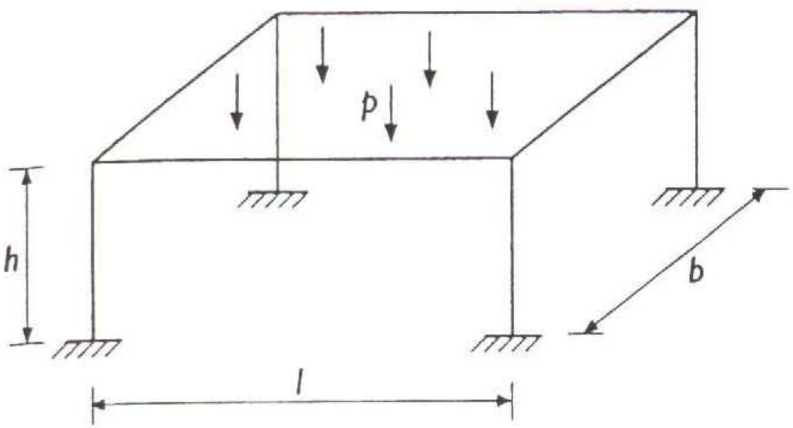
Column moment: $$M_c = \frac{\nu EI}{h^2} * S_d = \frac{\nu EI}{h^2} * \frac{PS_a}{\omega_o^2}$$

where, $\nu = 3$ for hinged-end, $\nu = 6$ for fixed-end columns.

Spectral 'static equivalence' approach (exact - !) is not a fully static analysis approach (false - X).



(a) Problem description



$h = 6\text{ m}$

$l = 12\text{ m}$

$b = 6\text{ m}$

$p = 5\text{ kN/m}^2$

$E = 35,000\text{ MPa}$

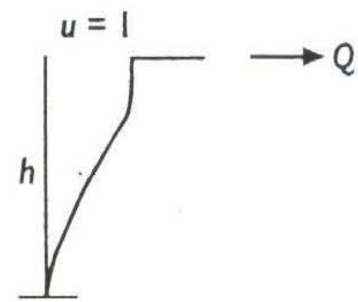
$I = 0.00213\text{ m}^4$

Column cross-section: $0.4 \times 0.4\text{ m}$

Column stiffness computation:

$$Q = 12\,EI/h^3$$

$$k = 4Q = 48EI/h^3 = 16,600\text{ kN/m}$$



Mass computation:

$$M = (plb)/g = 36.7\text{ kN sec}^2/\text{m}$$

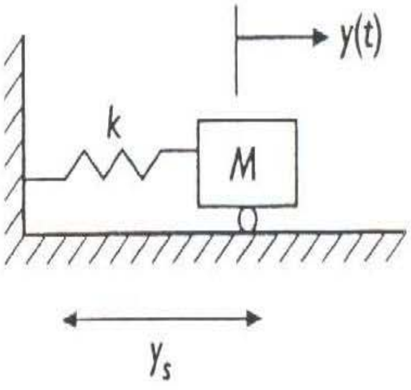
Damping coefficient:

$$\zeta = c/c_{cr} = 10\% = 0.1$$

(b) SDOF system model

$$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{16,600}{36.7}} = 21.3 \text{ rad/sec}$$

$$T = 2\pi/\omega = 0.30 \text{ sec}, \qquad f = 1/T = 3.38 \text{ Hz}$$



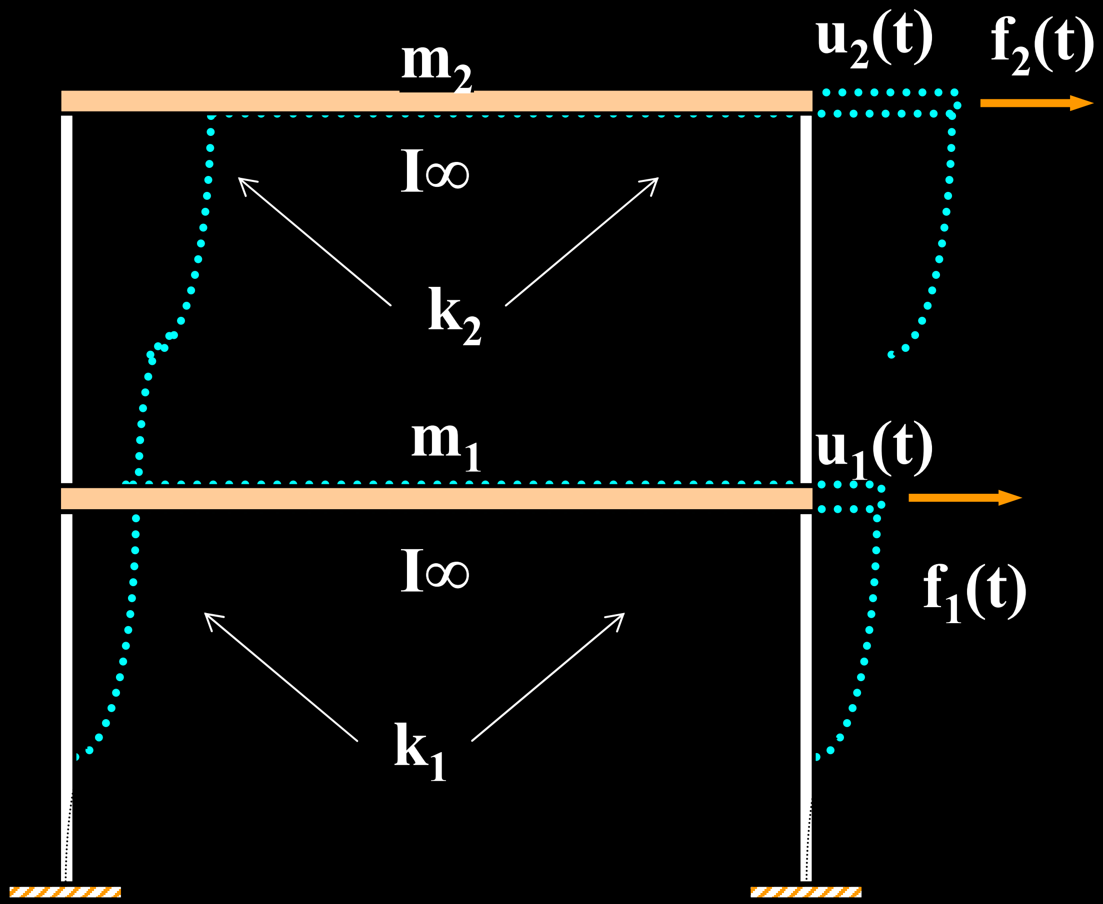
(c) Response spectrum computations

From the triple Kalamata 1986 earthquake response spectrum given in Figure 2.29, we have:

- maximum relative displacement is $u = y - y_s = 1.8$ cm;
- maximum velocity is $y = 35$ cm/sec;
- maximum acceleration is $\ddot{y} = 0.7\,g = 6.87\ \text{m/sec}^2$;
- maximum column shear is $V = (ku)/4 = 16,600(0.018)/4 = 74.7$ kN; and
- maximum column shear stress is $\tau = V/A = 74.7/(0.4^2) = 467\ \text{kN/m}^2$

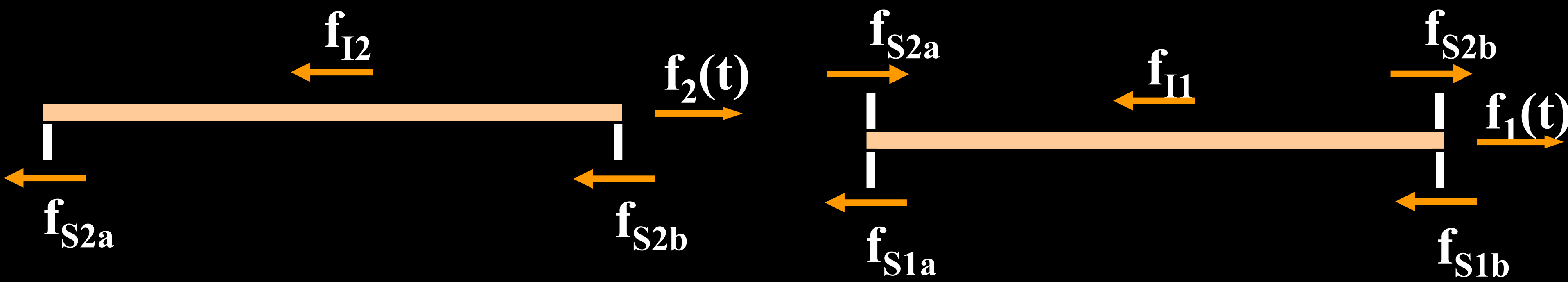
Two degree of freedom (2-dof) system

Rigid beams
Massless columns
Zero damping



Two storey shear-frame

Dynamic equilibrium



f_{Ij} = inertia force j = $m_j * \ddot{u}_j$

$f_{Sj} = f_{Sja} + f_{Sjb} = k_j*(u_j - u_i)$ = restoring force due to columns connecting levels j-1 and j.

$$f_{I2} + f_{S21} = f_2(t) \rightarrow m_2\, \ddot{u}_2 + k_2\,(u_2 - u_1) = f_2(t)$$

$$f_{I1} + f_{S12} + f_{S10} = f_1(t) \rightarrow m_1\, \ddot{u}_1 + k_2\,(u_1 - u_2) + k_1\, u_1 = f_1(t)$$

System of coupled differential equations

Matrix notation

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F}(t)$$

$$\mathbf{U} = \mathbf{U}(t) = \textbf{displacement vector} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\mathbf{M} = \textbf{mass matrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\mathbf{K} = \textbf{stiffness matrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\mathbf{F}(t) = \textbf{force vector} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

Free vibration of undamped 2-dof system

$$M*U'' + K*U = 0$$

$$U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \varphi_1 \cos(\omega t - \theta) \\ \varphi_2 \cos(\omega t - \theta) \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \cos(\omega t-\theta) =$$

$$\Phi \cos(\omega t-\theta)$$

$$\ddot{U}(t) = \begin{bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{bmatrix} = \begin{bmatrix} -\omega^2\varphi_1 \cos(\omega t - \theta) \\ -\omega^2\varphi_2 \cos(\omega t - \theta) \end{bmatrix} = -\omega_2\, \Phi \cos(\omega t-\theta)$$

$$M\,[-\omega^2\, \Phi \cos(\omega t-\theta)] + K\,[\Phi \cos(\omega t-\theta)] = [0] \rightarrow$$
$$\{K - \omega^2 M\}\, \Phi \cos(\omega t-\theta) = [0]$$

Unknowns are the amplitude vector Φ and the frequency of free oscillation ω.

$$\{K - \omega^2 M\}\, \Phi \cos(\omega t-\theta) = [0]$$

Should be valid for any time instant → zero determinant

$$\left|K - \omega^2 M\right| = [0] \rightarrow \begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix} = [0] \rightarrow$$

$$\omega^4 (m_1 m_2) - \omega^2 \{(k_1+k_2)m_2 + k_2 m_1\} + k_1 k_2 = 0$$

This is the frequency equation. Setting $\omega^2 = \lambda$, we get two solutions for λ and hence, two frequency values for free vibration $\lambda_1 = \omega_1^2$ and $\lambda_2 = \omega_2^2$.

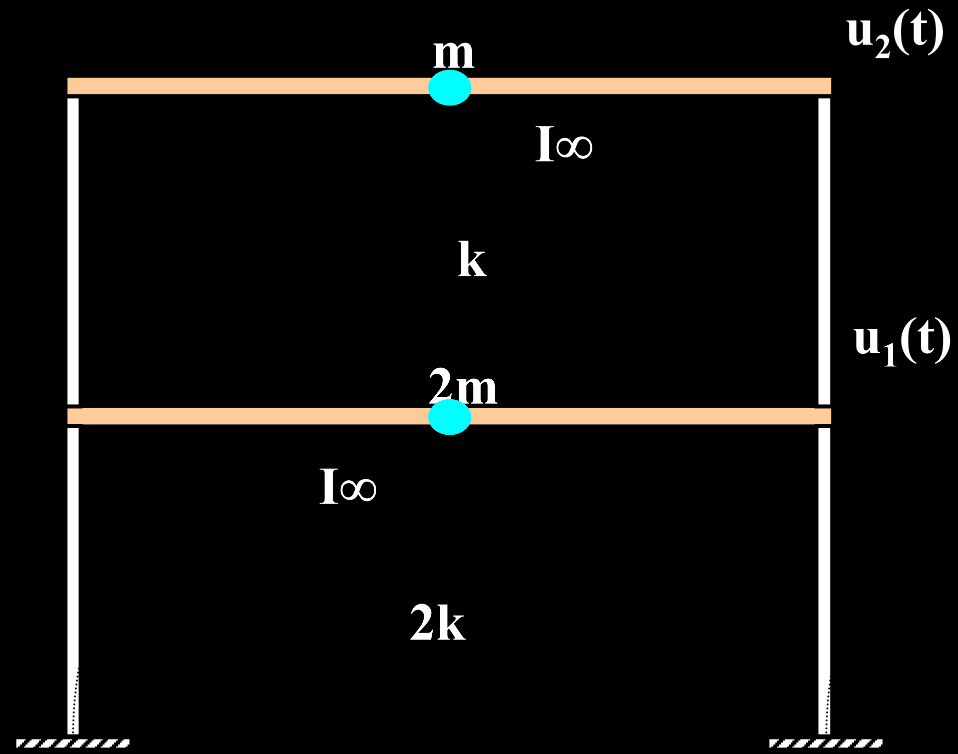
Therefore, a 2-dof system exhibits 2 natural frequencies, ω_1 and ω_2.

Substituting ω_1 and ω_2 back into the matrix equation, the two corresponding amplitude vectors (eigenvectors) can be evaluated.

$$\{K - \omega_j^2 M\} \Phi_j \cos(\omega_j t - \theta) = [0] \rightarrow \{K - \omega_j^2 M\} \Phi_j = [0]$$

The eigenvalue problem does not fix the absolute amplitude of the vectors Φ_j , but only the shape of the vector (relative values of displacement)

Example



$$\left.\begin{aligned}2m\ddot{u}_1 + 2ku_1 + k(u_1 - u_2) = 0\\ m\ddot{u}_2 + k(u_2 - u_1) = 0\end{aligned}\right\} \rightarrow \quad M\ddot{U} + KU = 0$$

$$\text{όπου}\ M = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix},\ K = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}\ \text{και}\ U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Natural frequencies determination

$$|K-\omega^2 M| = 0 \rightarrow \begin{vmatrix} 3k-2\omega^2 m & -k \\ -k & k-\omega^2 m \end{vmatrix} = 0 \rightarrow$$

$$2\omega^4 m^2 - 5\omega^2 km + 2k^2 = 0$$

Roots of quadratic equation $\omega_1^2 = k/2m$ και $\omega_2^2 = 2k/m$, with corresponding natural periods

$$T_1 = 2\pi/\omega_1 = \pi\sqrt{\frac{8m}{k}}, \qquad T_2 = 2\pi/\omega_2 = \pi\sqrt{\frac{2m}{k}}$$

Modal shapes calculation →

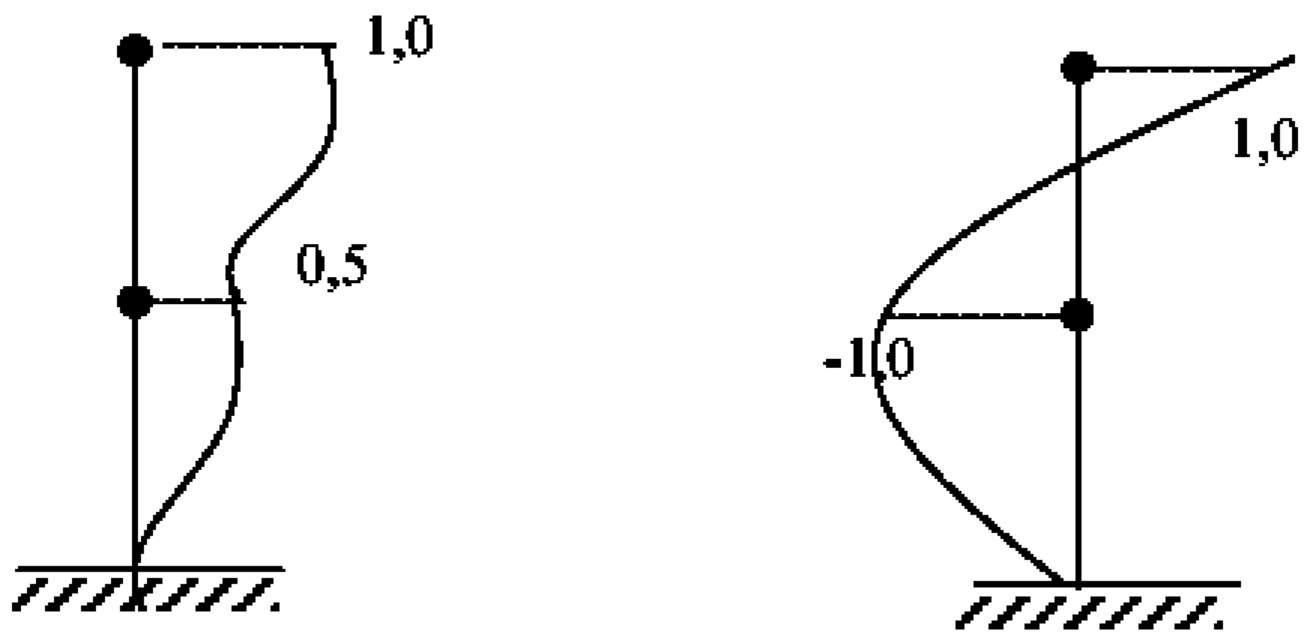
Eigenvectors $\Phi_1 = \begin{bmatrix} \varphi_{11} \\ \varphi_{21} \end{bmatrix}$ and $\Phi_2 = \begin{bmatrix} \varphi_{12} \\ \varphi_{22} \end{bmatrix}$, are computed as:

$$\omega_1^2 = k/2m \rightarrow \begin{bmatrix} 2k & -k \\ -k & k/2 \end{bmatrix} \begin{bmatrix} \varphi_{11} \\ \varphi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 2\varphi_{11} = \varphi_{21}$$

$$\omega_2^2 = 2k/m \rightarrow \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{bmatrix} \varphi_{12} \\ \varphi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \varphi_{12} = -\varphi_{22}$$

Setting (arbitrarily) $\varphi_{21} = \varphi_{22} = 1.0$, we get:

$$\Phi_1 = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \kappa\alpha\iota \ \Phi = \begin{bmatrix} 0.5 & -1 \\ 1 & 1 \end{bmatrix}$$


1,0
0,5
1,0
-1,0

Orthogonality of modes

Eigenvectors are orthogonal with respect to mass and stiffness matrices.

$$\Phi_j^T M \Phi_k = 0 \text{ and } \Phi_j^T K \Phi_k = 0, \ \gamma\iota\alpha\ j \neq k$$

Modal analysis

Set $$U(t) = \sum_{j=1}^{2} \Phi_j q_j(t) = \Phi\, Q(t)$$

Substitute to the matrix equation of motion:

$$M\ddot{U} + K\,U = [0] \rightarrow M\,\Phi\,\ddot{Q}(t) + K\,\Phi Q(t) = [0]$$

Pre-multiply all terms with Φ^T :

$$\Phi^T M\Phi\ddot{Q}(t) + \Phi^T K\Phi\, Q(t) = [0] \rightarrow M^*\ddot{Q}(t) + K^*\, Q(t) = [0]$$

The transformed matrix equation of free vibration, reads:

$$\mathbf{M^*}\ddot{Q}(t) + \mathbf{K^*\ Q(t) = [0]}$$

Due to orthogonality property the new matrices M* and K* are diagonal.

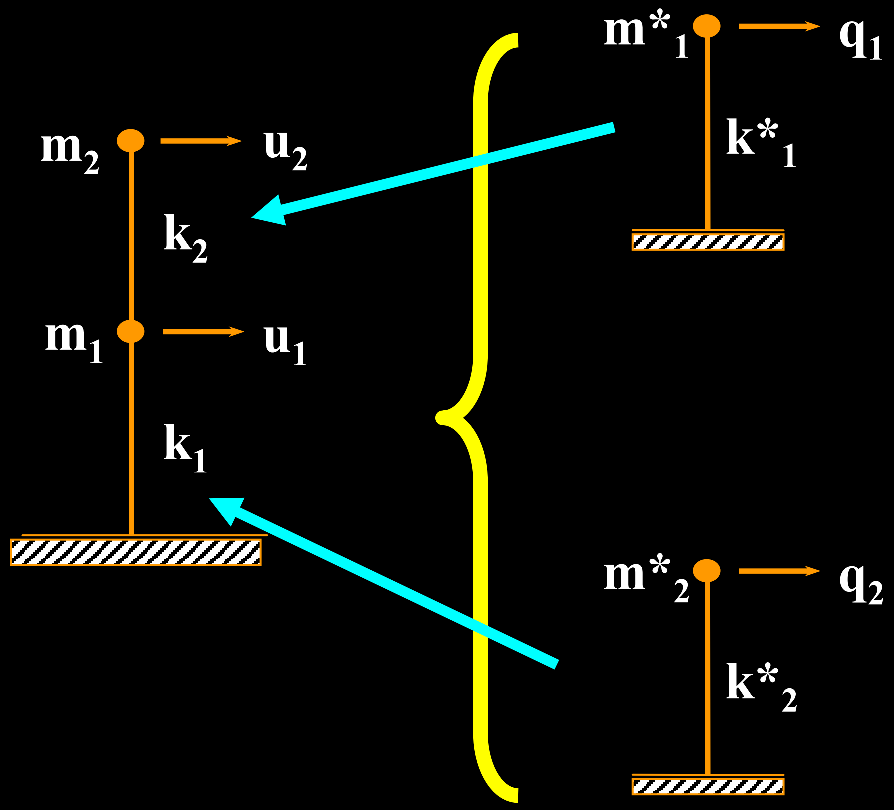
$$\mathbf{M^* = \text{generalized mass matrix}} = \begin{bmatrix} m_1^* & 0 \\ 0 & m_2^* \end{bmatrix}$$

$$\mathbf{K^* = \text{generalized stiffness matrix}} = \begin{bmatrix} k_1^* & 0 \\ 0 & k_2^* \end{bmatrix}$$

Therefore, the original matrix equation is transformed into a set of uncoupled sdof free vibration equations of the form (for j = 1, 2):

$$m_j^*\,\ddot{q}_j(t) + k_j^*\,\mathbf{q_j(t) = 0} \rightarrow \ddot{q}_j(t) + \omega_j^2\,\mathbf{q_j(t) = 0}$$

Modal decoupling



$$u_2(t) = u_{21}(t) + u_{22}(t) =$$
$$\varphi_{21}\, q_1(t) + \varphi_{22}\, q_2(t)$$

Forced vibration of a damped multi degree of freedom (mdof) system

Original (coupled) equation of motion:

$$\mathbf{M}\,\ddot{\mathrm{U}} + \mathbf{C}\,\dot{\mathrm{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F}(t)$$

Modal (decoupled) equation of motion:

$$\Phi^T M \Phi\,\ddot{Q} + \Phi^T C \Phi\,\dot{Q} + \Phi^T K \Phi\,Q = \Phi^T\,F(t) \rightarrow$$

$$M^*\,\ddot{Q} + C^*\,\dot{Q} + K^*\,Q = F^*(t)$$

where, C* = generalized damping matrix and F* = generalized force vector.

To ensure diagonalization of C*, here the assumption is made that the damping matrix of the original system C can be expressed as $C = \alpha \cdot M + \beta \cdot K$

Typical generalized (sdof) equation of motion:

$$m_j^* \ddot{q}_j + c_j^* \dot{q}_j + k_j^* \mathbf{q_j} = f_j^*(t) \rightarrow \ddot{q}_j + 2\boldsymbol{\xi_j \omega_j} \dot{q}_j + \boldsymbol{\omega_j^2 q_j} = \frac{f_j^*(t)}{m_j^*} = \tilde{f}_j(t)$$

To be solved within the framework of sdof theory (1st part of presentation).

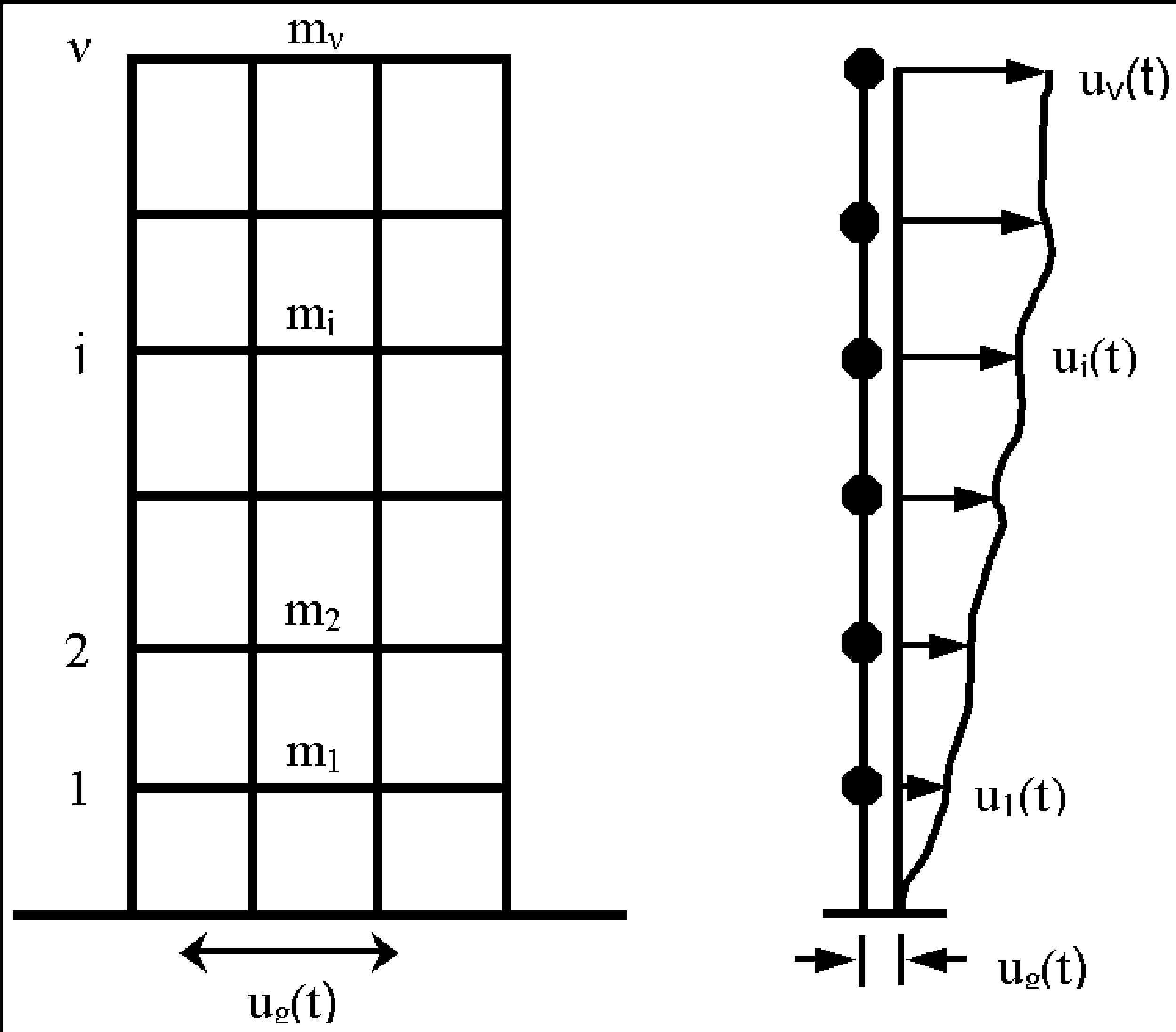
Following the determination of generalized vector Q, the original response vector U is computed as

$$\mathbf{U(t) = \Phi\ Q(t)} = \sum_{j=1}^{\nu} \Phi_j q_j(t)$$

The contribution of first modes are much more important than the contribution of higher modes.

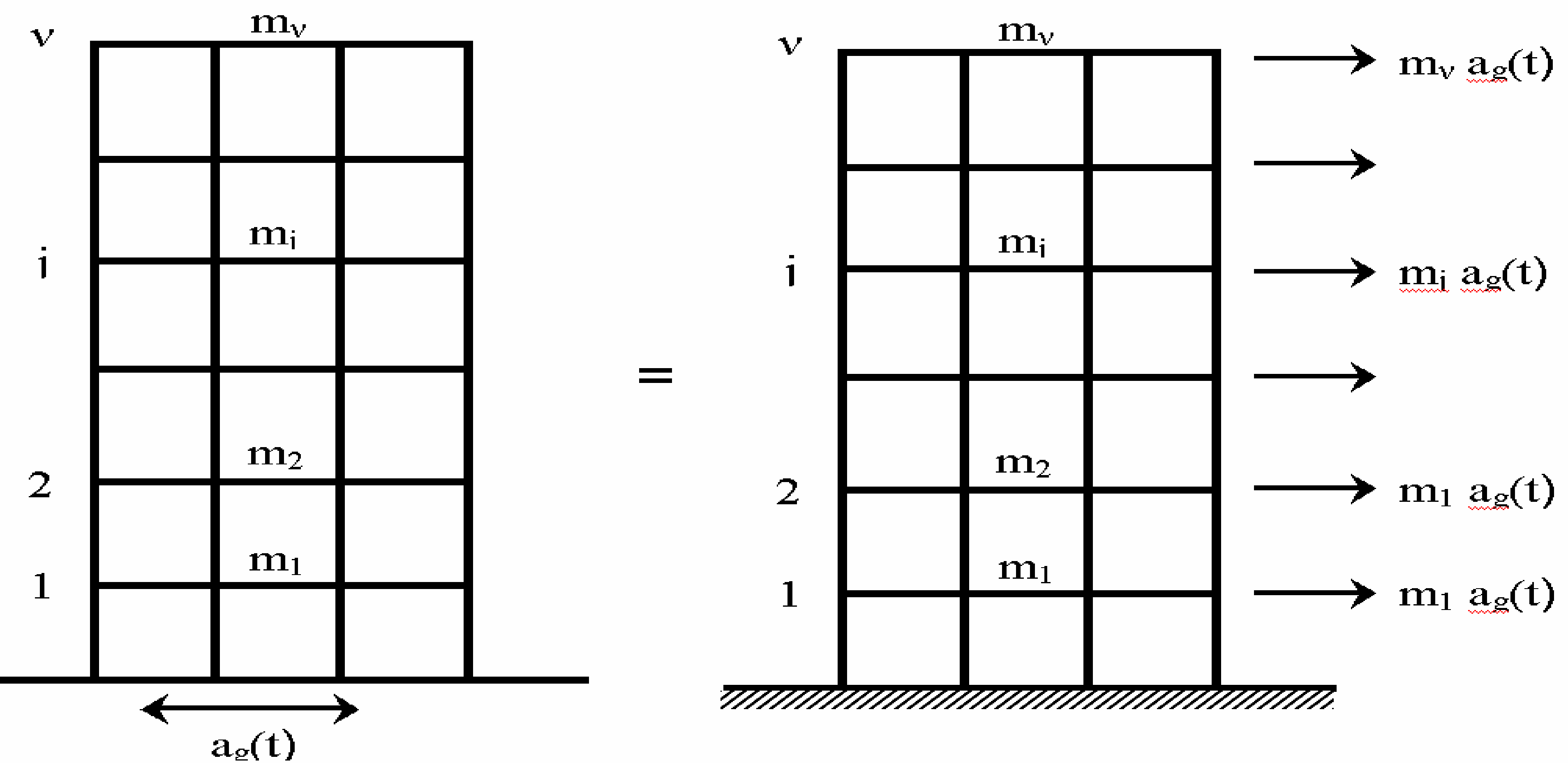
Earthquake excitation of mdof systems (Response spectrum analysis)

v-storey shear plane frame under ground motion $u_g(t)$



The total displacement vector $U_t(t)$, is composed by the relative displacement vector U(t) and the ground motion.

$$U_t(t) = U(t) + [1]u_g(t)$$



The matrix equation of motion of the original system is:

$$\mathbf{M}\,\ddot{U} + \mathbf{C}\,\dot{U} + \mathbf{K}\,\mathbf{U} = -\,\mathbf{M}\,[1]\,a_g(t) = F_g(t)$$

Firstly we compute ω_j και Φ_j, and then we proceed to the modal transformation

$$\Phi^T M \Phi\,\ddot{Q} + \Phi^T C \Phi\,\dot{Q} + \Phi^T K \Phi\,Q = \Phi^T F_g(t) \rightarrow$$

$$M^*\,\ddot{Q} + C^*\,\dot{Q} + K^*\,Q = F^*(t)$$

Here, the generalized force vector is

$$F^*(t) = \Phi^T F_g(t) = -\,\Phi^T M\,[1]\,a_g(t)$$

The sdof generalized equations are

$$\ddot{q}_j + 2\xi_j\omega_j\dot{q}_j + \omega_j^2 q_j = \frac{f_j^*(t)}{m_j^*} = -a_g(t)\frac{\sum_{k=1}^{\nu} m_k\varphi_{kj}}{\sum_{k=1}^{\nu} m_k\varphi_{kj}^2} = -\Gamma_j\, a_g(t)$$

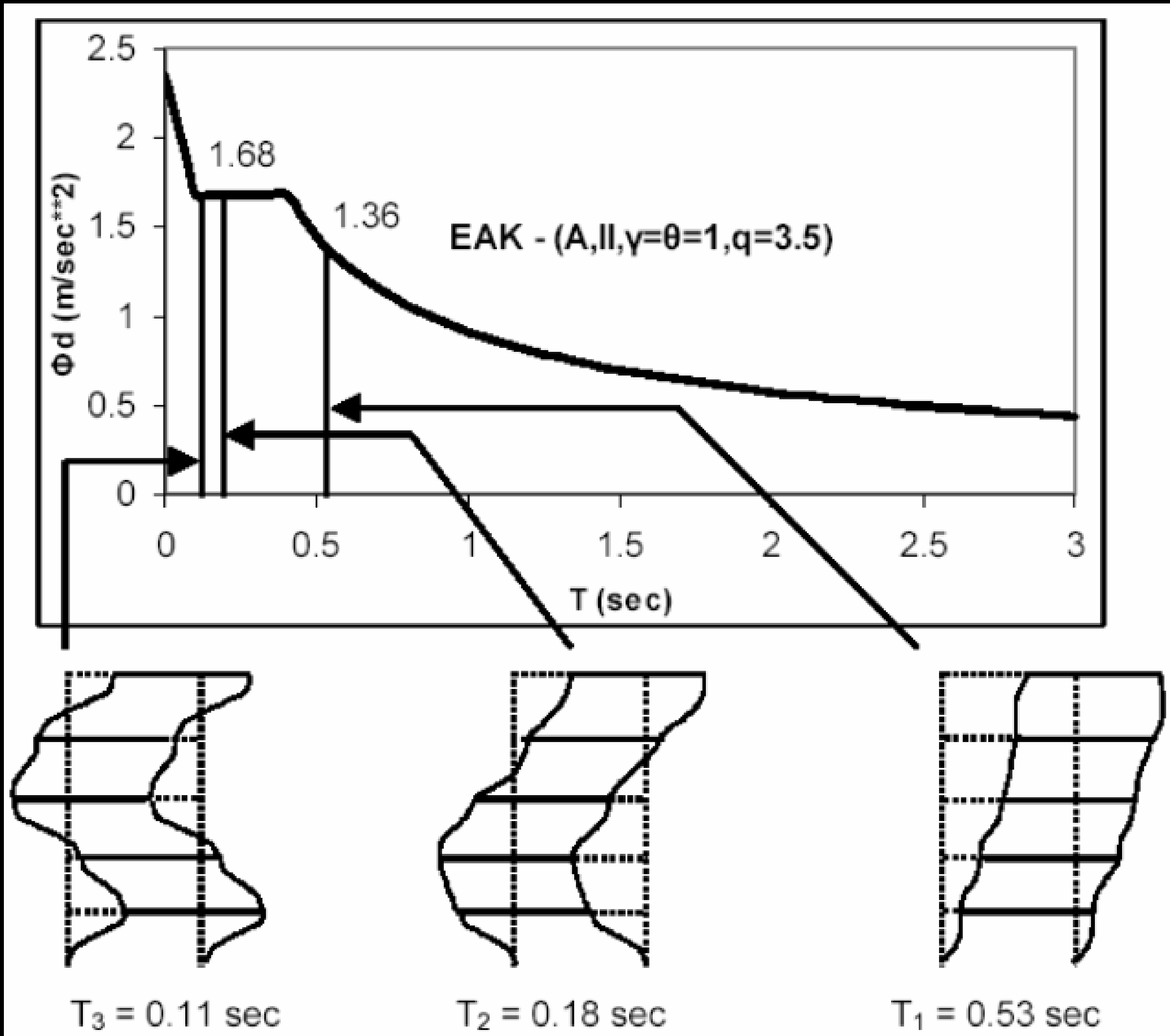
The generalized force parameter Γ_j is known as modal participation factor.

If the seismic action is expressed via the standard response or design spectra, the corresponding spectral values of the generalized response q_j, are

$$S_{d,j} = \Gamma_j\, S_d(T_j,\xi_j), \quad S_{v,j} = \Gamma_j\, S_v(T_j,\xi_j), \quad S_{a,j} = \Gamma_j\, S_a(T_j,\xi_j)$$

Example of utilization of Greek Design spectrum (EAK) for the estimation of modal spectral accelerations of a mdof frame

The ordinates of the design spectrum should be multiplied by the corresponding modal participation factors Γ_j.



The problem of combination of modal peak values

The following decomposition of physical response u_j in terms of generalized (modal) components q_k is valid for any instant of time.

$$u_j(t) = \sum_{k=1}^{\nu} \varphi_{jk} q_k(t) = \sum_{k=1}^{\nu} u_{jk}(t)$$

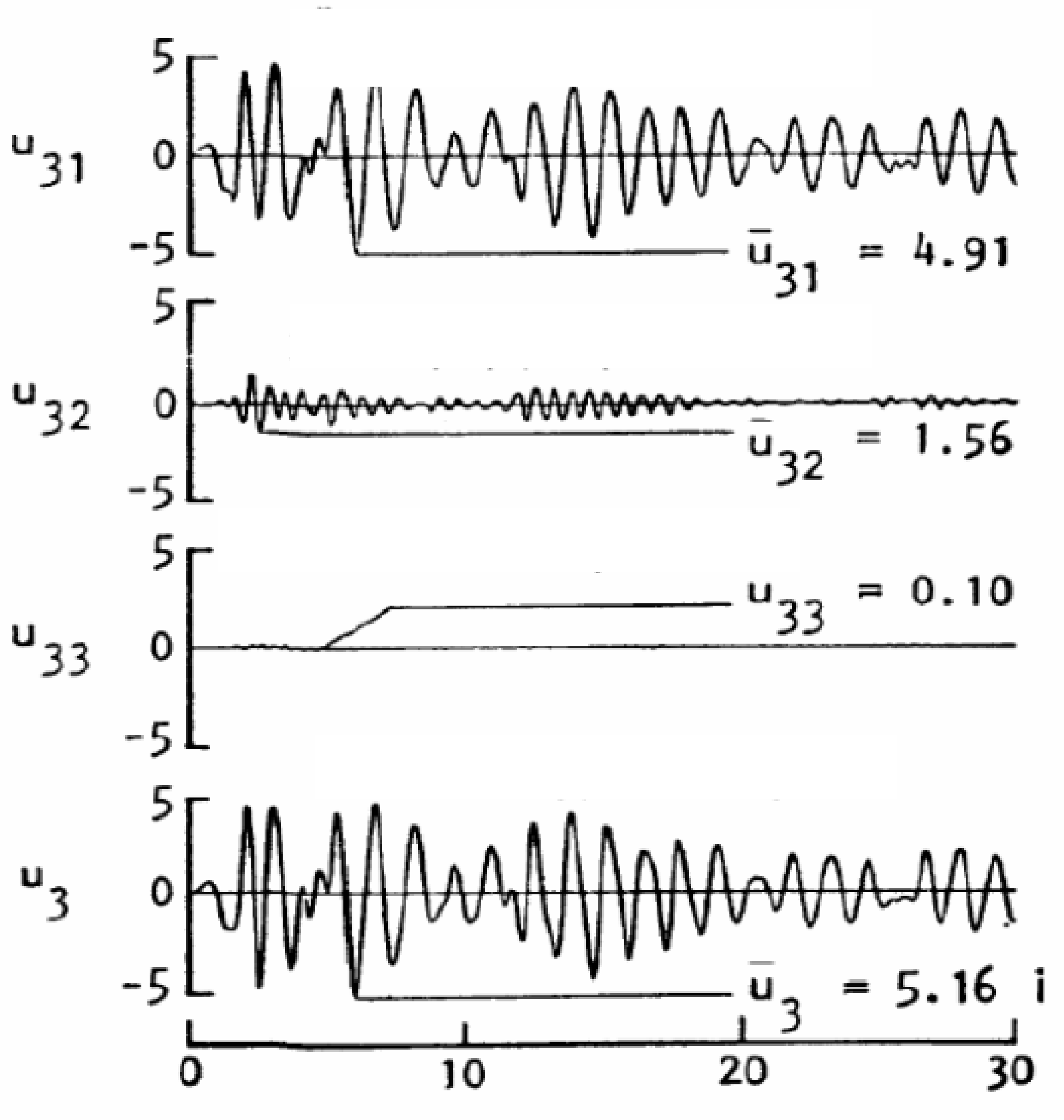
where $u_{jk}(t)$ is the 'contribution' of k modal component $q_k(t)$ to the response of the j degree of freedom $u_j(t)$ of the original system.

However, if only spectral (peak) modal response quantities are available

$$\bar{u}_{jk} = \varphi_{j,k} \, \Gamma_k \, S_d(T_k, \xi_k)$$

these do not occur at the same time and hence, cannot be added to obtain the peak value of $u_j(t)$

Modal contributions
$u_{3k}(t)$ (for k = 1,2,3) to the response of the top floor of a 3-storey frame

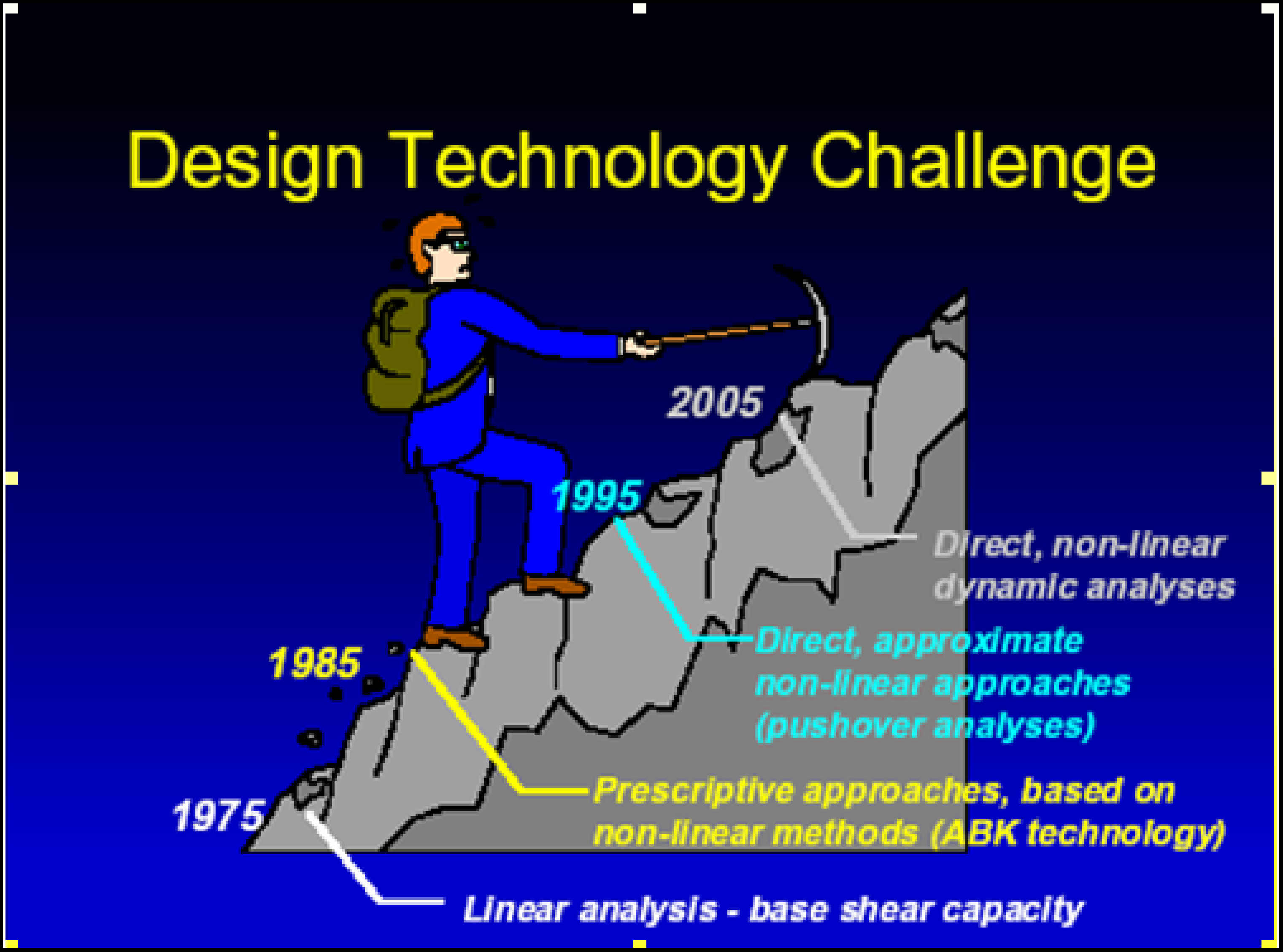


$(\bar{u}_{31} + \bar{u}_{32} + \bar{u}_{33}) =$

$(4.91 + 1.56 + 0.10) =$
$6.57 > 5.16$

Modal combination rule SRSS

$$\sqrt{\bar{u}_{31}^2 + \bar{u}_{32}^2 + \bar{u}_{33}^2} = \sqrt{4.91^2 + 1.56^2 + 0.10^2} = \mathbf{5.15 \approx 5.16}$$


Design Technology Challenge
2005
1995
1985
1975
Direct, non-linear dynamic analyses
Direct, approximate non-linear approaches (pushover analyses)
Prescriptive approaches, based on non-linear methods (ABK technology)
Linear analysis - base shear capacity

THE END